Distinguishing Symplectic Blowups of $\mathbb{CP}^2$

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Recall $\mathbb{CP}^n$.

$$\begin{array}{ccc} S^{2n+1} & \overset{i}{\hookrightarrow} & \mathbb{C}^{n+1} \\ \Big\downarrow{\scriptstyle \pi} & & \\ \mathbb{CP}^n & & \pi^*\omega_{\mathrm{FS}} = i^*\omega_{\mathbb{C}^n} \end{array}$$

$n = 1: \ \mathbb{CP}^1 = S^2, \ \text{area} = \pi.$

$$\text{size} := \frac{1}{2\pi}\ \text{area}$$

Symplectic blowup of size $\delta = \frac{\mathbf{r}^2}{\mathbf{2}}$:

$$B^{2n}(r+\epsilon) \xhookrightarrow[\text{symplectic}]{} M$$

$$\text{The blowup} \ = \ \left(M \smallsetminus B^{2n}(r)\right)/\sim$$

$$= \ \left(M \smallsetminus \overline{B^{2n}(r)}\right) \ \sqcup \ \underbrace{\partial B^{2n}(r)/S^1}_{\cong (\mathbb{CP}^{n-1},\, r^2\omega_{\text{FS}})}$$

Blowups of $\mathbb{CP}^2$:

Fix $p_1, \ldots, p_k \in \mathbb{CP}^2$.
$M_k :=$ complex blowup of $\mathbb{CP}^2$ at $p_1, \ldots, p_k$.

$$H_2(M_k) = \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \ldots \oplus \mathbb{Z}E_k$$

$L \cong \mathbb{CP}^1 \subset \mathbb{CP}^2$
$E_1, \ldots, E_k$ exceptional divisors

Uniqueness of symplectic blowups of $\mathbb{CP}^2$:

$\mathcal{SF}_{M_k} := \{$ symplectic forms ω on M_k such that each of $L, E_1, \ldots, E_k$ contains an embedded ω-symplectic sphere $\}$.

Let $\omega, \omega' \in \mathcal{SF}_{M_k}$.

1. $\exists$ diffeomorphism $\psi \colon M_k \to M_k$, trivial on H_2, such that $\psi^*\omega$ and ω' are homotopic in $\mathcal{SF}_{M_k}$.
2. If $[\omega] = [\omega']$, $\exists$ diffeomorphism $\psi \colon M_k \to M_k$, trivial on H_2, such that $\psi^*\omega = \omega'$.

(McDuff)

Classical invariants of $(\mathbf{M_k}, \omega)$ for $\omega \in \mathcal{SF}_{\mathbf{M_k}}$:

Write $\omega = \omega_{\lambda;\delta_1,\ldots,\delta_k}$ where

$$\lambda = \frac{1}{2\pi}\int_L \omega$$

$$\delta_j = \frac{1}{2\pi}\int_{E_j} \omega \quad j = 1, \ldots, k$$

$$\omega\text{–volume} = \lambda^2 - \delta_1^2 - \ldots - \delta_k^2 \quad \text{times } \frac{(2\pi)^2}{2!}$$

$$\langle \omega, c_1 \rangle = 3\lambda - \delta_1 - \ldots - \delta_k \quad \text{times } 2\pi$$

Question: which symplectic blowups of $\mathbb{CP}^2$ are symplectomorphic?

Example:

	λ	δ_1	δ_2	δ_3
ω	15	9	5	4
ω'	12	6	2	1
ω''	11	4	1	1

$$(M_k, \omega) \cong (M_k, \omega') \not\cong (M_k, \omega'')$$

same classical invariants: $\quad \text{volume} = \dfrac{(2\pi)^2}{2!} \cdot 103$

$$\langle \omega, c_1 \rangle = (2\pi) \cdot 27$$

Answer:

Definition: $(\lambda; \delta_1, \ldots, \delta_k)$ *reduced* if

$$\begin{cases} \delta_1 \geq \ldots \geq \delta_k, \text{ and} \\ \delta_1 + \delta_2 + \delta_3 \leq \lambda. \end{cases}$$

Theorem: Given $(M_k, \omega_{\lambda'; \delta'_1, \ldots, \delta'_k})$, there exists a unique reduced $(\lambda; \delta_1, \ldots, \delta_k)$ such that

$$(M_k, \omega_{\lambda'; \delta'_1, \ldots, \delta'_k}) \text{ is symplectomorphic to } (M_k, \omega_{\lambda; \delta_1, \ldots, \delta_k}).$$

Recall: $(\lambda; \delta_1, \ldots, \delta_k)$ is **reduced** if $\delta_1 \geq \ldots \geq \delta_k$ and $\delta_1 + \delta_2 + \delta_3 \leq \lambda$.

Existence of reduced form:

- Take $\omega = \omega_{\lambda;\delta_1,\ldots,\delta_k}$. If $(\lambda; \delta_1, \ldots, \delta_k)$ not reduced:

$$\text{defect} := \delta_1 + \delta_2 + \delta_3 - \lambda$$

- Cremona:

$$\begin{aligned} \lambda &\rightsquigarrow \lambda - \text{defect} & & \\ \delta_j &\rightsquigarrow \delta_j - \text{defect} & & \text{for } j = 1, 2, 3 \\ \delta_j &\rightsquigarrow \delta_j & & \text{for } j = 4, \ldots, k \end{aligned}$$

- Repeat.

Exceptional classes with minimal area - definition.

Given $(M_k, \omega_{\lambda;\delta_1,\dots,\delta_k})$, with $v = (\lambda; \delta_1, \dots, \delta_k)$ reduced.

Exceptional classes $:= \{E \in H_2(M_k) \mid E$ contains an embedded ω-symplectic sphere and $E \cdot E = -1\}$.

$\mathcal{E}_{\min}(M_k, \omega) := \{$ exceptional classes of smallest ω-area $\}$.

Recall: $(\lambda; \delta_1, \ldots, \delta_k)$ is **reduced** if $\delta_1 \geq \ldots \geq \delta_k$ and $\delta_1 + \delta_2 + \delta_3 \leq \lambda$.

Cases:

(1) $\delta_1 \leq \lambda/3$, and

- (1a) $\delta_k \leq \lambda/3$
- (1b) $\delta_k = \lambda/3$

(2) $\delta_1 > \lambda/3$ and $\delta_2 \leq (\lambda - \delta_1)/2$, and

- (2a) $\delta_k < (\lambda - \delta_1)/2$
- (2b) $\delta_k = (\lambda - \delta_1)/2$

(3) $\delta_1 > \lambda/3$ and $\delta_2 > (\lambda - \delta_1)/2$, and

- (3a) $\lambda - \delta_1 - \delta_2 > \delta_k$
- (3b) $\lambda - \delta_1 - \delta_2 = \delta_k$

Exceptional classes with minimal area - theorem.

$v = (\lambda; \delta_1, \ldots, \delta_k)$ reduced, $\mathcal{E}_{\min} := \mathcal{E}_{\min}(M_k, \omega_{\lambda;\delta_1,\ldots,\delta_k})$.

$$\delta_1 \geq \ldots \geq \delta_j > \delta_{j+1} = \ldots = \delta_k.$$

Cases (1a), (2a), (3a): $\mathcal{E}_{\min} = \{E_{j+1}, \ldots, E_k\}$.

Case (1b): $v = (\lambda; \lambda/3, \ldots, \lambda/3)$. $k \leq 8$.
$\mathcal{E}_{\min} =$ all exceptional classes. – listed by Demazure.

Case (2b): $v = (\lambda; \delta_1, \lambda_F/2, \ldots, \lambda_F/2)$.
$\mathcal{E}_{\min} = \{E_2, \ldots, E_k, E_{12}, \ldots, E_{1k}\}$ where $E_{1j} = L - E_1 - E_j$.

Case (3b): $\delta_2 > \delta_3 = \ldots = \delta_k = \lambda - \delta_1 - \delta_2$.
$\mathcal{E}_{\min} = \{E_{12}, E_3, \ldots, E_k\}$.

Technical Lemma used to identify $\mathcal{E}_{\min}$:

Assume

$k \geq 3$;

$(M, \omega) = (M_k, \omega_{\lambda;\delta_1,\ldots,\delta_k})$ for $(\lambda; \delta_1, \ldots, \delta_k)$ reduced;

J compatible almost complex structure on (M, ω).

Assume

$A \in H_2(M)$ is represented by a J-sphere; $c_1(TM)(A) \geq 1$.

Then

$$\int_A \omega \geq \delta_k.$$

Uniqueness of reduced form - sketch of proof

Given $\omega = \omega_{\lambda;\delta_1,\dots,\delta_k}$ and $\omega' = \omega_{\lambda';\delta'_1,\dots,\delta'_k}$.
$(\lambda;\delta_1,\dots,\delta_k)$, $(\lambda';\delta'_1,\dots,\delta'_k)$ reduced.

A symplectomorphism $\psi\colon (M_k,\omega) \to (M_k,\omega')$
takes $\mathcal{E}_{\min} := \mathcal{E}_{\min}(M_k,\omega)$ to ${\mathcal{E}_{\min}}' := \mathcal{E}_{\min}(M_k,\omega')$.

- If not every two classes in $\mathcal{E}_{\min}$ are disjoint:
  - same for ${\mathcal{E}_{\min}}'$.
  - count maximal disjoint subset $\Rightarrow$ both are case (1b) or both are case (2b).
  - same classical invariants $\Rightarrow (\lambda;\delta_1,\dots,\delta_k) = (\lambda';\delta'_1,\dots,\delta'_k)$.
- If every two classes in $\mathcal{E}_{\min}$ are disjoint:
  - same for ${\mathcal{E}_{\min}}'$.
  - blow down along disjoint embedded symplectic spheres in $\mathcal{E}_{\min}, {\mathcal{E}_{\min}}'$.
  - apply induction hypothesis.
- Base case: works when $k = 2$. (use blowdown along E_{12}.)

Thank you.