New Topological Quantum Field Theories from Orbifolds

Ernesto Lupercio[1]

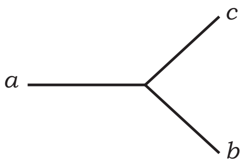
August 5, 2013

[1] CINVESTAV-IPN, Mexico

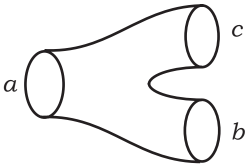
Acknowledgments:

Joint work with: Ana Gonzalez (Uruguay), Carlos Segovia (Heilderberg) and Bernardo Uribe (Bonn and Bogotá).

2d TQFT (closed sector)



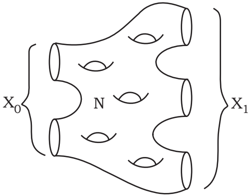
Particle interaction



String interaction

Definition inspired by String Theory (Witten).

From fields to functors



Let $H_{X_i} := \mathrm{Maps}(\mathcal{F}(X_i), \mathbb{C})$. Then we can write a linear operator of the form:

$$Z_N \colon H_{X_0} \longrightarrow H_{X_1},$$

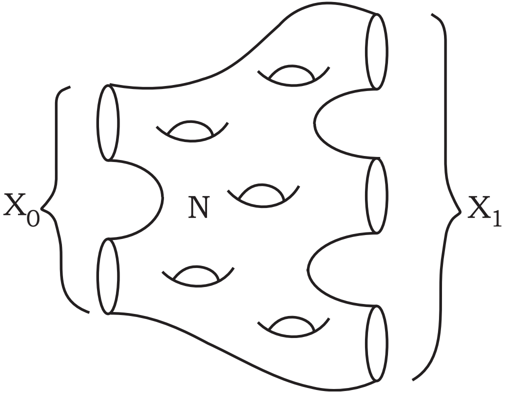
by the formula:

$$(Z_N(\Psi))(\psi_1) = \int_{\mathcal{F}(X_0)} K(\psi_1, \psi_0)\Psi(\psi_0)\mathcal{D}\psi_0,$$

where the kernel K is given by

$$K(\phi_1, \phi_2) = \int_{\phi \in \mathcal{F}(N),\ \phi|_{X_i} = \psi_i} e^{-iS(\phi)}\mathcal{D}\phi.$$

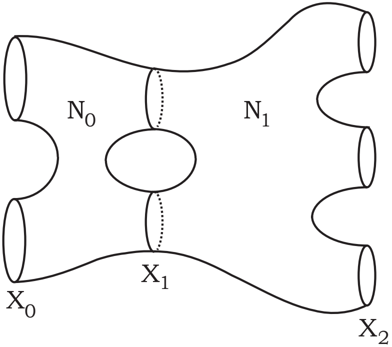
2d TQFT definition



$$Z_N \colon H_{X_0} \longrightarrow H_{X_1}$$

$$Z_N \colon H_{X_{01}} \otimes H_{X_{02}} \longrightarrow H_{X_{11}} \otimes H_{X_{12}} \otimes H_{X_{13}},$$

2d TQFT definition

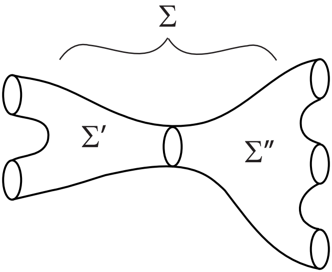


We require that

$$Z_N = Z_{N_1} \circ Z_{N_0},$$

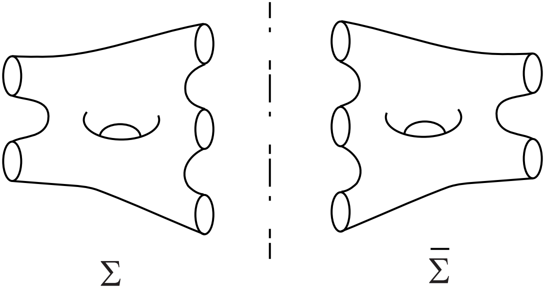
$$Z_N \colon H_{X_{01}} \otimes H_{X_{02}} \longrightarrow H_{X_{11}} \otimes H_{X_{12}} \otimes H_{X_{13}}.$$

2d TQFT definition



$$Z_\Sigma = Z_{\Sigma''} \circ Z_{\Sigma'}$$

2d TQFT definition



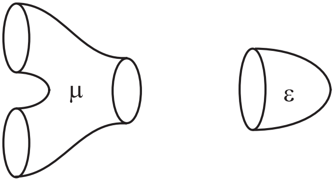
$$Z_{\bar{\Sigma}} = Z_{\Sigma}^{*}$$

2d TQFT definition

$$Z_{S^1 \times I} = id_A$$

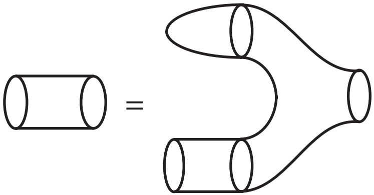
The Frobenius Algebra

The structure of a TQFT on A automatically endows A with the structure of a Frobenius algebra, where we have the product as the operator induced by the pair of pants and the trace as the operator induced by the right sided cap:

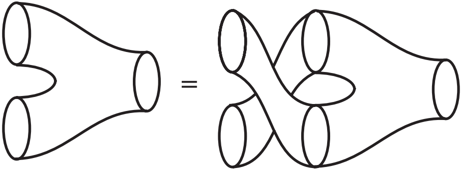


$$A \otimes A \xrightarrow{\mu} A \qquad A \xrightarrow{\varepsilon} \mathbb{k}$$

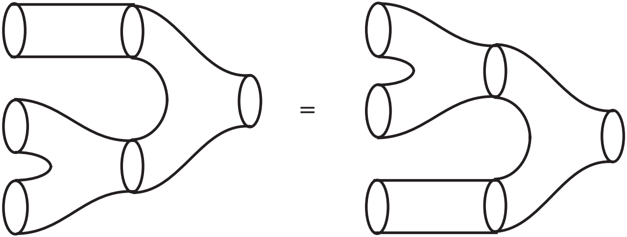
Unit Axiom



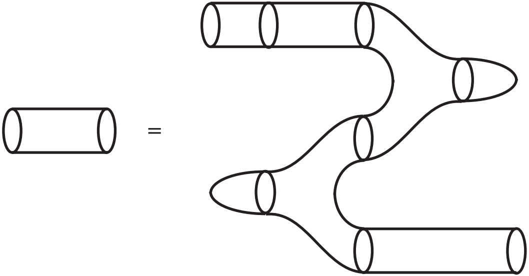
Commutativity



Associativity



Non-degeneracy



The classification theorem

Theorem

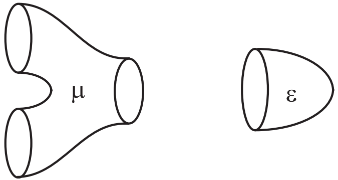
There is a canonical equivalence of categories:

$$\text{2D-TFT}_{\mathbb{k}} \simeq \text{cFA}_{\mathbb{k}}$$

where $\text{cFA}_{\mathbb{k}}$ *is the category of commutative Frobenius algebras.*

The Case of Positive Boundaries

We define the category $nCob^+$ by considering its object to be oriented *non-empty* closed smooth $(n-1)$-dimensional manifolds, and the morphisms are the oriented smooth n-dimensional manifolds(n-cobordism). Notice we do not allow the empty manifold to be the in-boundary nor the out-boundary. We always have components on both sides so, in the following picture, the first cobordism m is allowed while the second ε is *forbidden*:



The classification theorem

Theorem (Gonzalez, -, Segovia, Uribe)

There is a one-to-one correspondence between nearly Frobenius algebras and (1+1)-dimensional positive boundary topological quantum field theories.

Definition of a Nearly-Frobenius Algebra.

Definition

A *nearly Frobenius algebra* A is an algebra together with a commutative coassociative comultiplication $\Delta : A \longrightarrow A \otimes A$ such that Δ is an A-bimodule morphism.

What this means explicitly is that whenever $\Delta(b) = \sum_i b_i \otimes b_i'$ we have in turn that the following equation holds:

$$\Delta(ab) = \sum_i (a \cdot b_i) \otimes c_i.$$

Also if $\Delta(a) = \sum_i a_i \otimes a_i'$, then

$$\Delta(ab) = \sum_i a_i \otimes (a_i' \cdot b)$$

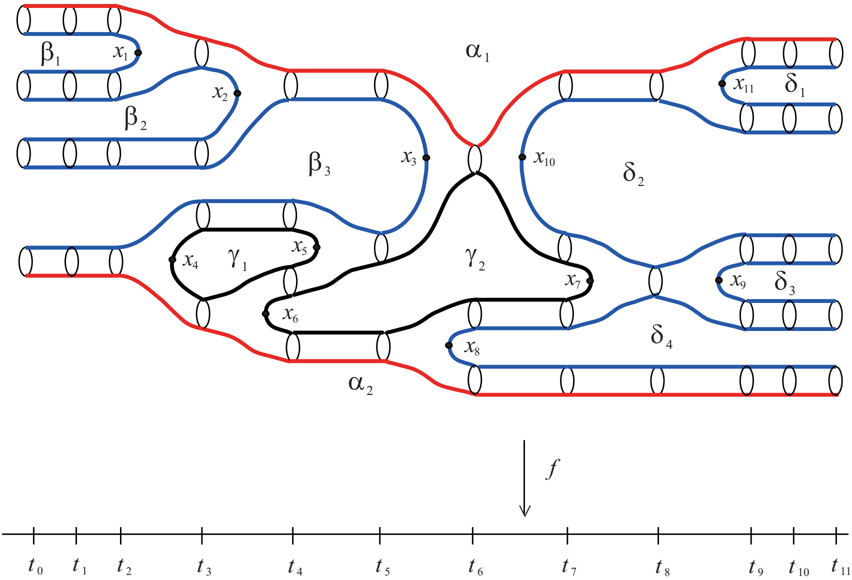
.

We write these identities more compactly as follows:

$$\Delta(ab) = a\Delta(b) = \Delta(a)b \qquad (1)$$

and we call these equations the Abrams' condition.

Non-degeneracy



Examples

Example

Let A be the truncated polynomial algebra in one variable $\Bbbk[x]/x^{n+1}$. We will determine all the nearly-Frobenius structures on A. Then, the pair (A, Δ) is a nearly-Frobenius algebra. In particular, we have that the coproduct Δ is a linear combination of the coproducts Δ_k defined by:

$$\Delta_k(x^l) = \sum_{i+j=n+k+l} x^i \otimes x^j, \quad \text{for } k \in \{0, \dots, n\}$$

that is $\Delta = \sum_{k=0}^{n} a_k \Delta_k$ where $a_k \in \Bbbk$ for all $k \in \{1, \dots, n\}$. Note that Δ_0 is the Frobenis coproduct of A where the trace map $\varepsilon : A \to \mathbb{C}$ is given by $\varepsilon(x^i) = \delta_{i,n}$. The other coproducts, Δ_k $k \neq 0$, do not come from a Frobenius algebra structure. That is, it does not exist a trace map $\varepsilon : A \to A \otimes A$ such that $(A, \Delta_k, \varepsilon)$ is a Frobenius algebra for $k = 1, \dots, n$.

Examples

Example

Let A be the algebra $\mathbb{C}[[x, x^{-1}]]$ of formal Laurent series. Consider the coproducts given by:

$$\Delta_j(x^i) = \sum_{k+l=i+j} x^k \otimes x^l.$$

These coproducts define nearly Frobenius structures that do not come from a Frobenius structure.

Examples

Example

The Poincaré algebra $A := H^*(M)$ of a non-compact manifold M is a nearly Frobenius algebra.

Examples

Consider the diagram:

$$\begin{array}{ccc} M & \xrightarrow{\Delta} & M \times M \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle 1\times\Delta} \\ M \times M & \xrightarrow[\Delta\times 1]{} & M \times M \times M \end{array}$$

Using transversality, we have that:

$$(\Delta \times 1)^*(1 \times \Delta)^! = \Delta^! \Delta^*,$$

where $\Delta^* : \mathrm{H}^*(M) \otimes \mathrm{H}^*(M) = \mathrm{H}^*(M \times M) \to \mathrm{H}^*(M)$, and $\Delta^! : \mathrm{H}^*(M) \to \mathrm{H}^*(M) \otimes \mathrm{H}^*(M)$ is the Gysin map. Therefore,

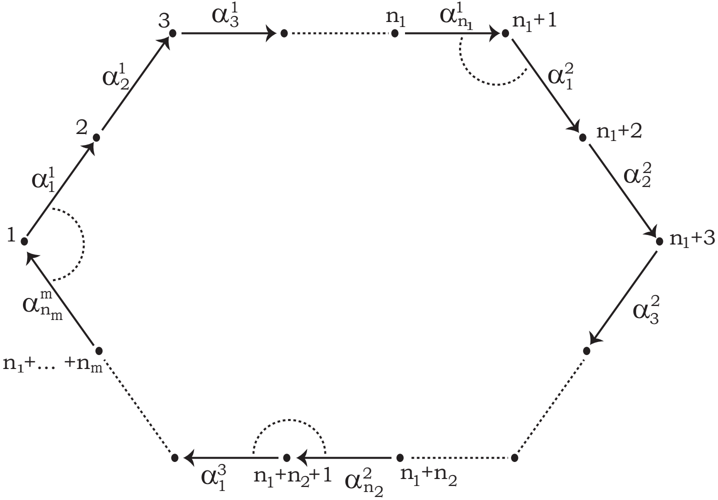
$$\left(\Delta^* \otimes 1\right)\left(1 \otimes \Delta^!\right) = \Delta^! \Delta^*.$$

Then $\mathrm{H}^*(M)$ is an algebra with a coproduct which is a module homomorphism.

Examples

Theorem (Ana Gonzalez,Artenstein, Lanzilotta)

The path algebra A associated to the cyclic quiver Q



Examples

Theorem (Ana Gonzalez,Artenstein, Lanzilotta)

with m maximal paths of length n_i, $i = 1, \ldots, m$ admits R nearly-Frobenius structures, where

$$R = m + \sum_{i=1}^{m} n_i n_{i+1}$$

whit $n_{m+1} = m_1$.

The Moduli Space of Nearly Frobenius Structures of a Fixed Algebra A

Theorem

Let A be a fixed $\mathbb{k}$-algebra. Then the set of nearly Frobenius coproducts of A making it into a nearly Frobenius algebra is a $\mathbb{k}$-vector space.

Definition

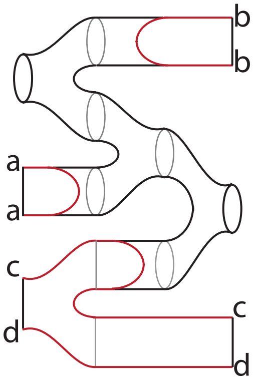
The Frobenius space associated to an algebra A is the vector of all the possible co-products Δ that make it into a nearly Frobenius algebra. Its dimension over $\mathbb{k}$ is called the *Frobenius* dimension of A.

Finite groups

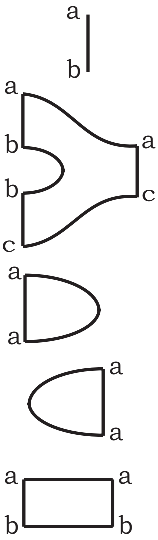
If $char(\mathbb{k})$ does not divide the order of G, then $\mathbb{k}[G]$ is a nearly-Frobenius algebra.

In the case of semi-simple algebras the Frobenius space of A is a vector space of dimension equal to the dimension of A, and that it has a one dimensional subspace (minus the origin) of *bona fide* Frobenius structures.

(Non-compact) Calabi-Yau Categories



Data for Calabi-Yau Categories



$$\Longrightarrow \quad \mathcal{O}_{ab}$$

$$\Longrightarrow \mathbf{F}(\Sigma): \mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \longrightarrow \mathcal{O}_{ac}$$

$$\Longrightarrow \quad \theta_a : \mathcal{O}_{aa} \longrightarrow k$$

$$\Longrightarrow \quad \mathbf{F}(\Sigma): \begin{array}{ccc} k & \longrightarrow & \mathcal{O}_{aa} \\ 1 & \longmapsto & 1_a \end{array}$$

$$\Longrightarrow \quad \mathbf{F}(\Sigma) = \mathrm{Id}: \mathcal{O}_{ab} \longrightarrow \mathcal{O}_{ab}$$

Figure: Basic data for the open theory.

Perfect Pairings

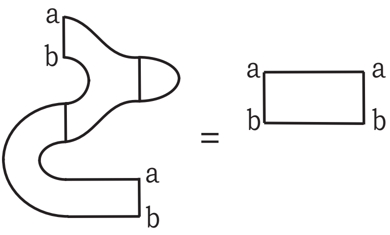
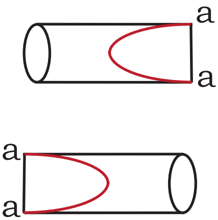


Figure: Assuming that the strip corresponds to the identity morphism, we must have perfect pairings.

Open-Closed Transitions



$$\iota_a \colon A \longrightarrow \mathcal{O}_{aa}$$

$$\iota^a \colon \mathcal{O}_{aa} \longrightarrow A$$

Figure: Two ways of representing open to closed and closed to open transitions.

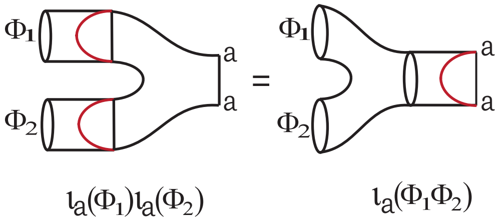


Figure: ι_a is a homomorphism.

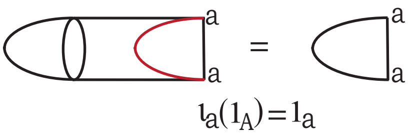


Figure: ι_a preserves the identity.

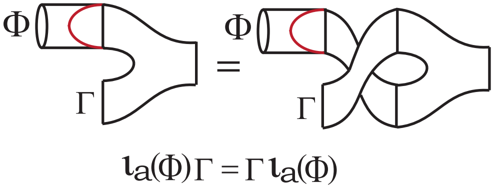


Figure: ι_a maps into the center of $\mathcal{O}_{aa}$.

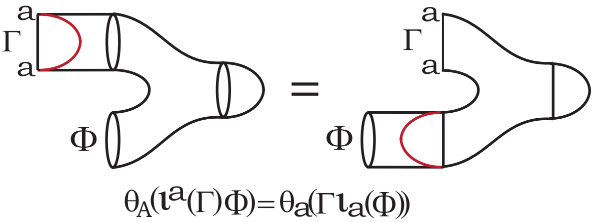


Figure: ι^a is the adjoint of ι_a.

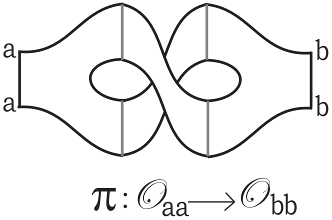


Figure: The double-twist diagram defines the map $\pi^a_b : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$.

The Cardy Condition

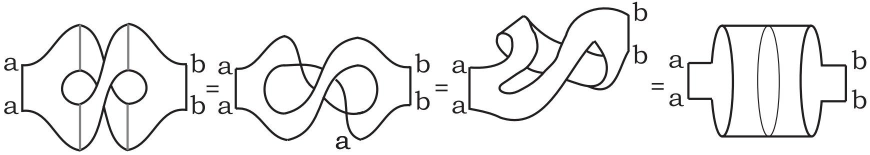


Figure: The Cardy-condition is expressing the factorization of the double-twist diagram in the closed string channel.

$$\pi^a_b = \iota_b \circ \iota^a. \tag{2}$$

Example: Representations of a Finite Group G

Consider a finite group G. Where the category $\mathcal{B}$ is the category $\mathcal{R}ep(G)$ of finite dimensional representations of G. If $E \in \mathrm{Obj}\big(\mathcal{R}ep(G)\big)$, the trace $\theta_E : \mathcal{O}_{EE} \to \mathbb{C}$ takes $\psi : E \to E$ to $\frac{1}{|G|}\,\mathrm{tr}(\psi)$.
The algebra A is the center of the group algebra $\mathbb{C}[G]$ such that:

$$\iota_E : Z\big(\mathbb{C}[G]\big) \to \mathcal{O}_{EE},$$
$$\sum_g \alpha_g g \mapsto \sum_g \alpha_g \rho_g$$

$$\iota^E : \mathcal{O}_{EE} \to Z\big(\mathbb{C}[G]\big),$$
$$\psi : E \to E \mapsto \sum_g \mathrm{tr}\big(\psi g|_E\big) g^{-1}$$

and the trace

$$\theta_{Z\left(\mathbb{C}[G]\right)} : Z\big(\mathbb{C}[G]\big) \to \mathbb{C}$$
$$\sum_g \alpha_g g \mapsto \frac{\alpha_1}{|G|}.$$

Definition of NEARLY Calabi-Yau Categories

1. $(\mathcal{A}, \Delta_{\mathcal{A}}, 1_{\mathcal{A}})$ is a commutative non compact Frobenius algebra.

2 $\mathcal{O}_{ab}$ is a collection of vector spaces for $a, b \in \mathcal{B}$.

2a. There is a family of associative linear maps:

$$\eta^{b}_{ac} : \mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac} \tag{3}$$

2b. There is a family of co-associative linear maps

$$\Delta^{c}_{ab} : \mathcal{O}_{ab} \to \mathcal{O}_{ac} \otimes \mathcal{O}_{cb}.$$

2c. Moreover, Δ^{c}_{ab} is a morphism of $\mathcal{O}_{da} \times \mathcal{O}_{be}$-bimodule, i.e. the diagrams

$$\begin{array}{ccc} \mathcal{O}_{da} \otimes \mathcal{O}_{ab} & \xrightarrow{\eta^{a}_{db}} & \mathcal{O}_{db} \\ {\scriptstyle 1\otimes\Delta^{c}_{ab}}\downarrow & & \downarrow{\scriptstyle \Delta^{c}_{db}} \\ \mathcal{O}_{da} \otimes \mathcal{O}_{ac} \otimes \mathcal{O}_{cb} & \xrightarrow[\eta^{a}_{dc}\otimes 1]{} & \mathcal{O}_{dc} \otimes \mathcal{O}_{cb} \end{array} \qquad \begin{array}{ccc} \mathcal{O}_{ab} \otimes \mathcal{O}_{bb} & \xrightarrow{\eta^{b}_{ae}} & \mathcal{O}_{ae} \\ {\scriptstyle \Delta^{c}_{ab}\otimes 1}\downarrow & & \downarrow{\scriptstyle \Delta^{c}_{ae}} \\ \mathcal{O}_{ac} \otimes \mathcal{O}_{cb} \otimes \mathcal{O}_{be} & \xrightarrow[1\otimes\eta^{b}_{ce}]{} & \mathcal{O}_{ac} \otimes \mathcal{O}_{ce} \end{array}$$

commute.

3. There are linear maps:

$$\iota_a : \mathcal{A} \to \mathcal{O}_{aa}, \iota^a : \mathcal{O}_{aa} \to \mathcal{A} \tag{4}$$

such that

3a. ι_a is an algebra homomorphism:

$$\iota_a(\phi_1\phi_2) = \iota_a(\phi_1)\iota_a(\phi_2) \tag{5}$$

3b. The identity is preserved

$$\iota_a(1_{\mathcal{A}}) = 1_a \tag{6}$$

3c. Moreover, ι_a is central in the sense that:

$$\iota_a(\phi)\psi = \psi\iota_b(\phi) \tag{7}$$

for all $\phi \in \mathcal{A}$ and $\psi \in \mathcal{O}_{ab}$.

3d. The *Cardy conditions*: we define the map $\pi^a_b : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$ as follows. Since $\mathcal{O}_{ab}$ and $\mathcal{O}_{ba}$ are in duality (using θ_a or θ_b), if we let ψ_μ be a basis for $\mathcal{O}_{ba}$, then there is a dual basis ψ^μ for $\mathcal{O}_{ab}$. Then we set:

$$\pi^a_b(\psi) = \sum_\mu \psi_\mu \psi \psi^\mu, \tag{8}$$

and the *Cardy condition* is

$$\pi^a_b = \iota_b \circ \iota^a. \tag{9}$$

String Topology

Let M be a smooth, orientable manifold of dimension n. The *space of free loop space* is:

$$\mathcal{L}M = \{\alpha : S^1 \to M\},$$

where every loop is assumed piecewise smooth.
Chas and Sullivan in [CS] proved the next result.

String Topology

Theorem (Chas and Sullivan, 1999)

Let M be a compact, closed, smooth, orientable manifold of dimension d. There is a commutative and associative product

$$\mathrm{H}_p(\mathcal{L}M) \otimes \mathrm{H}_q(\mathcal{L}M) \to \mathrm{H}_{p+q-d}(\mathcal{L}M)$$

- *making $\mathbb{H}_*(\mathcal{L}M) := \mathrm{H}_{*+d}(\mathcal{L}M)$ an associtive, commutative graded algebra and*
- *compatible with the intersection product on $\mathrm{H}_*(M)$, i.e., the following diagram commutes.*

$$\begin{array}{ccc}
\mathrm{H}_p(\mathcal{L}M) \otimes \mathrm{H}_q(\mathcal{L}M) & \longrightarrow & \mathrm{H}_{p+q-d}(\mathcal{L}M) \\
\Big\downarrow \scriptstyle{\mathrm{ev}_* \otimes \mathrm{ev}_*} & & \Big\downarrow \scriptstyle{\mathrm{ev}_*} \\
\mathrm{H}_p\, M \otimes \mathrm{H}_q\, M & \longrightarrow & \mathrm{H}_{p+q-d}\, M
\end{array}$$

String Topology

Theorem (Gonzalez, -, Segovia, Uribe)

String Topology can be extended to a Nearly Calabi-Yau category. Let $\mathcal{B}$ be the category of D-branes, the objects of this category are a collection of submanifolds of M,

$$\mathrm{Obj}(\mathcal{B}) = \left\{ D_i \subset M : \ D_i \ \textit{is a submanifold of } M, i \in I \right\}.$$

Now, we consider the path spaces:

$$\mathcal{P}_M(D_i, D_j) = \{\gamma : [0,1] \to M \ \textit{picewise smooth} \ : \gamma(0) \in D_i, \gamma(1) \in D_j\}$$

Then, the morphisms of the category $\mathcal{B}$ are:

$$\mathrm{Hom}_{\mathcal{B}}(D_i, D_j) = \mathrm{H}_*(\mathcal{P}_M(D_i, D_j)),$$

for $D_i, D_j \in \mathrm{Obj}(\mathcal{B})$.

String Topology

Relates results

- Cohen-Godin: Closed sector, nearly Frobenius algebra (different proof) [CG04].
- Baas-Cohen-Ramirez. Homotopy type of the category. [**?**]
- Blumberg-Cohen-Teleman. A derived version. [BCT09]

Nearly G-Calabi-Yau

As before, we define the notion of a G-open-closed theory with positive boundary as a G-open-closed theory but with the restriction that the morphisms have at least one outgoing boundary.
The algebraic characterization is the following:

1. A nearly G-Frobenius algebra associated to the circle.

Nearly G-Calabi-Yau

1. For each pair a, b of labels a vector space $\mathcal{O}_{ab}$ with a G-action

$$\rho : G \to \mathrm{Aut}(\mathcal{O}_{ab})$$

such that:

$$\rho_g(\eta^c_{ab}(\varphi_1 \otimes \varphi_2)) = \eta^c_{ab}(\rho_g(\varphi_1) \otimes \rho_g(\varphi_2)),$$
$$\Delta^c_{ab}(\rho_g(\varphi)) = (\rho_g \otimes \rho_g)\Delta^c_{ab}(\varphi),$$

for $\varphi_1 \in \mathcal{O}_{ac}$, $\varphi_2 \in \mathcal{O}_{cb}$, $\varphi \in \mathcal{O}_{ab}$ and $g \in G$. This conditions are represented in the figures 10 and 11.

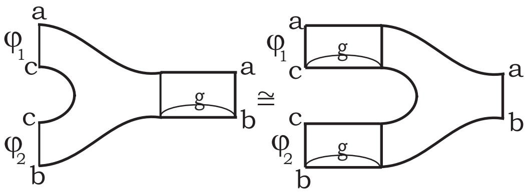


Figure: The product is a G-morphism with the diagonal action.

Nearly G-Calabi-Yau

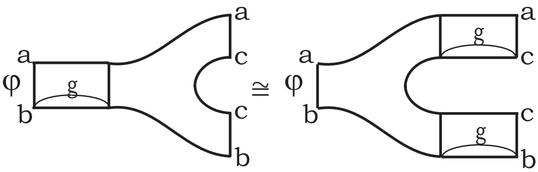


Figure: The coproduct is a G-morphism with the diagonal action.

Nearly G-Calabi-Yau

1. For every label a the vector space $\mathcal{O}_{aa}$ is non necessarily a commutative nearly Frobenius algebra.
2. There are also G-twisted (*equivariant*) open-closed transition maps:

$$\iota_{g,a} : \mathcal{C}_g \to \mathcal{O}_{aa},$$
$$\iota^{g,a} : \mathcal{O}_{aa} \to \mathcal{C}_g,$$

The map $\iota : \mathcal{C} \to \mathcal{O}$ is obtained by adding up ι_g, i.e. $\iota = \oplus_{g \in G} \iota_g$ is a ring homomorphisms, then:

$$\iota_{g_1}(\Phi_1)\iota_{g_2}(\Phi_2) = \iota_{g_2 g_1}(\Phi_2 \Phi_1),$$

with $\Phi_1 \in \mathcal{C}_{g_1}$ and $\Phi_2 \in \mathcal{C}_{g_2}$. Moreover $\iota_e(1_{\mathcal{C}}) = 1_{\mathcal{O}_{aa}}$. The G-twisted centrality condition is:

$$\iota_g(\Phi)(\rho_g \Psi) = \Psi \iota_g(\Phi),$$

where $\Phi \in \mathcal{C}_g$ y $\Psi \in \mathcal{O}_{aa}$.

Nearly G-Calabi-Yau

1. The G-twisted Cardy conditions. For each $g \in G$, we must have:

$$\pi^{a}_{g,b} = \iota_{g,b}\iota^{g,a}.$$

Hence, $\pi^{a}_{g,b}$ is defined by:

$$\pi^{a}_{g,b} := \eta^{a}_{bb} \circ \tau \circ (1 \otimes \rho_g) \circ \Delta^{b}_{aa} : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$$

where $\tau : \mathcal{O}_{ab} \otimes \mathcal{O}_{ba} \to \mathcal{O}_{ba} \otimes \mathcal{O}_{ab}$ is the transposition map (see figure 12).

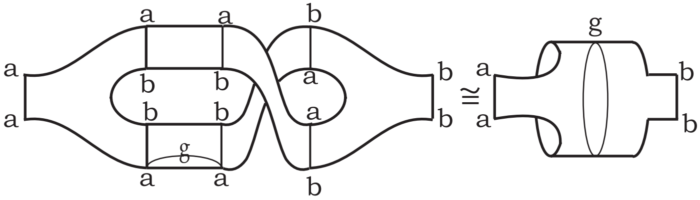


Figure: G-twisted Cardy condition.

Nearly G-Calabi-Yau

Theorem

The G-invariant part of a G-OC TFT with positive boundary is an OC-TFT with positive boundary.

Orbifold String Topology

Theorem (Gonzalez, -, Segovia, Uribe)

Orbifold String Topology can be extended to a G-Nearly Calabi-Yau category.

Virtual Orbifold Cohomology

Let S be a complex manifold and let S_1 and S_2 be closed submanifolds that intersect *cleanly*; that is, $U := S_1 \cap S_2$ is a submanifold of S and at each point x of U the tangent space of U is the intersection of the tangent spaces of S_1 and S_2. Let $E(S, S_1, S_2)$ be the *excess* bundle of the intersection, i.e., the vector bundle over U which is the quotient of the tangent bundle of S by the sum of the tangent bundles of S_1 and S_2 restricted to U. Thus $E(S, S_1, S_2) = 0$ if and only if S_1 and S_2 intersect transversally. In the Grothendieck group of vector bundles over U, the excess bundle becomes:

$$E(S, S_1, S_2) = T_S|_U + T_U - T_{S_1}|_U - T_{S_2}|_U.$$

Virtual Orbifold Cohomology

Denote by $e(S, S_1, S_2)$, the Euler class of $E(S, S_1, S_2)$ and by:

$$\begin{array}{ccc} U & \xrightarrow{i_1} & S_1 \\ {\scriptstyle i_2}\downarrow & \overset{h}{\searrow} & \downarrow{\scriptstyle j_1} \\ S_2 & \xrightarrow[j_2]{} & S \end{array} \tag{10}$$

the relevant inclusion maps. Then, for any cohomology class $\alpha \in H^*(S_1)$, the following *excess intersection formula* [Qui71, Prop. 3.3] holds in the cohomology ring of S_2:

$$j_2^* j_{1*}\alpha = i_{2*}\left(e(S, S_1, S_2) i_1^*(\alpha)\right). \tag{11}$$

Virtual Orbifold Cohomology

Consider the orbifold $[Y/G]$ where Y is an almost complex manifold and G acts preserving the almost complex structure. Define the groups:

$$H^*(Y,G) := \bigoplus_{g\in G} H^*(Y^g)\times\{g\}$$

where Y^g is the fixed point set of the element g. The group G acts in the natural way. Denote by $Y^{g,h} = Y^g \cap Y^h$ and suppose that, for every $g,h\in G$, we have cohomology classes $v(g,h)\in H^*(Y^{g,h})$, which are G-equivariant in the sense that $w^*v(k^{-1}gk, k^{-1}hk) = v(g,h)$ where $w: Y^{k^{-1}gk,k^{-1}hk}\to Y^{g,h}$ takes x to $w(x) := xk$. Define the map:

$$\begin{aligned} \times : H^*(Y^g)\times H^*(Y^h) &\to H^*(Y^{gh}) \\ (\alpha,\beta) &\mapsto i_*\left(\alpha|_{Y^{g.h}}\cdot\beta|_{Y^{g,h}}\cdot v(g,h)\right) \end{aligned}$$

where $i: Y^{g,h}\to Y^{gh}$ is the natural inclusion.

Virtual Orbifold Cohomology

Let us define now a degree shift σ on $H^*(Y, G)$. We will declare that the degree of a class $\alpha_g \in H^*(Y^g) \subset H^*(Y, G)[\sigma]$ is:

$$i + \sigma_g$$

where

$$\sigma_g := 2(\dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} Y^g),$$

and i is the ordinary degree of α_g. In this book, all dimensions and codimensions are complex. Virtual orbifold cohomology was introduced in [LUX07]. There it was shown that:

Virtual Orbifold Cohomology

Theorem

For the cohomology classes $v(g,h) = e(Y, Y^g, Y^h)$ *the map* $\times$ *defines an associative graded product on* $H^*_{\mathrm{virt}}(Y,G) := H^*(Y,G)[\sigma]$.

Virtual Orbifold Cohomology

Theorem (Gonzalez, -, Segovia, Uribe)

Virtual Orbifold Cohomology Extends to a Nearly Calabi-Yau category. The open part is the following: Let be $\mathcal{B} = \{X \subset M \colon G\text{-}invariant\}$ *such that, if* $X, Y \in \mathcal{B}$ *then* $TX|_{(X \cap Y)^g} \cong TY|_{(X \cap Y)^g}$ *for all* $g \in G$. *We define* $\mathrm{Hom}_{\mathcal{B}}(X, Y) = \mathrm{H}^*(X \cap Y)$, *for* $X, Y \in \mathcal{B}$.

Virtual Orbifold Cohomology

Theorem (Gonzalez, -, Segovia, Uribe)

Let $\left(\mathrm{H}^*_{\mathrm{virt}}(M;G),\mathcal{B}\right)$ *be the open closed virtual cohomology of* $[M/G]$*. If we change the correction clases of the open virtual coproduct and the closed map by*

$$E_\varepsilon(X,Y,Z)=\left(TM+T(X\cap Y\cap Z)-T(X\cap Z)+\varepsilon TY\right)|_{X\cap Y\cap Z}$$

and

$$F_g^\varepsilon=\left(TX^g+\varepsilon TX\right)|_{X^g}$$

we have a one parameter family of open closed TFT with positive boundary, where $\varepsilon\in\mathbb{C}$*.*

Chen-Ruan Cohomology

We will give now the definition of the Chen-Ruan cohomology following [CR04]. First, we need to define the degree shifting and the obstruction bundle for the Chen-Ruan theory.
The definition of the degree shifting is local so it is enough to define it in the case of a global quotient (cf. [FG03]).
Consider Y an almost complex G-manifold with G a finite group. Given $g \in G$ and $y \in Y^g$, we define $a(g, y)$ the *age* of g at y as follows. Diagonalize the action of g in $T_y Y$ to obtain:

$$g = \mathrm{diag}(\exp(2\pi i r_1), \ldots, \exp(2\pi i r_n)),$$

with $0 \leq r_i < 1$ and set:

$$a(g, y) := \sum_i r_i.$$

Chen-Ruan Cohomology

The age $a(g, y)$ only depends on the connected component Y_o^g of Y^g in which y lies. For this reason, we can simply write $a(g, Y_o^g)$ or even $a(g)$ when there is no confusion.
Note that the age has the following interesting property:

$$a(g, Y_o^g) + a(g^{-1}, Y_o^g) = \mathrm{codim}(Y_o^g, Y).$$

The *Chen-Ruan degree shifting number* is defined then as:

$$s_g := 2a(g).$$

As a rational vector space the Chen-Ruan orbifold cohomology is:

$$H^*_{\mathrm{CR}}(Y, G) := H^*(Y, G)[s] = \bigoplus_{g \in G} H^*(Y^g, \mathbb{C})[s_g]$$

or more generally:

$$H^*_{\mathrm{CR}}(\mathcal{G}) := H^*(I(\mathcal{G}))[s].$$

Chen-Ruan Cohomology

The definition of the obstruction bundle is modeled on the definition of the virtual fundamental class on the moduli of curves for quantum cohomology.

Let $\bar{\mathcal{M}}_3(\mathcal{G})$ be the moduli space of ghost representable orbifold morphisms f_y from $\mathbb{P}^1_3$ to $\mathcal{G}$, where $\mathrm{im}(f) = y \in \mathcal{G}_0$ and the marked orbifold Riemann surface $\mathbb{P}^1_3$ has three marked points, z_1, z_2, and z_3, with multiplicities m_1, m_2, and m_3, respectively. In [ALR07] they prove that

$$\bar{\mathcal{M}}_3(\mathcal{G}) = \mathcal{G}^2.$$

Let us fix a connected component $\mathcal{G}^2_o$ of $\mathcal{G}^2$.

To define the Chen-Ruan obstruction bundle $\mathcal{E}_o \to \mathcal{G}^2_o$ we consider the elliptic complex:

$$\bar{\partial}_y : \Omega^0(f_y^* T\mathcal{G}) \longrightarrow \Omega^{0,1}(f_y^* T\mathcal{G}).$$

Chen and Ruan proved that $\mathrm{coker}(\bar{\partial}_y)$ has constant dimension along components and forms an orbivector bundle $\mathcal{E}_o \to \mathcal{G}^2_o$.

Chen-Ruan Cohomology

The formula for the Chen-Ruan product is then (see Section **??**)

$$H^*_{\mathrm{CR}}(\mathcal{G}) \otimes H^*_{\mathrm{CR}}(\mathcal{G}) \longrightarrow H^*_{\mathrm{CR}}(\mathcal{G})$$

given by:

$$\alpha \star \beta := (e_{12})_*(e_1^*\alpha \cdot e_2^*\beta \cdot e(\mathcal{E})).$$

Chen-Ruan Cohomology

The following is a theorem of Chen and Ruan [CR04] (cf. [Kau03].)

Theorem

$(H^*_{\mathrm{CR}}(\mathcal{G}), \star)$ is a graded associative algebra, moreover it has a natural Frobenius algebra structure compatible with this product. In addition it is a G-Frobenius algebra in the global case.

Virtual Orbifold Cohomology and Chen-Ruan Cohomology

Corollary

In either one of the following cases the hypothesis of theorem **??** *are satisfied:*

- $\mathcal{X} = [M/G]$ *is hyperkahler (e.g. M hyperkahler and G acting by hyperkahler isomorphisms.)*
- $\mathcal{X} = \mathcal{Y} \times \mathcal{Y}$ *for a complex orbifold* $\mathcal{Y}$.
- $\mathcal{X} = T\mathcal{Y}$ *for a complex orbifold* $\mathcal{Y}$.

and hence, we have in those cases:

$$j^* : H^*_{CR}(T^*M, G) \xrightarrow{\cong} H^*_{virt}(M, G),$$

and

$$\mathcal{K}(T^*M, G; \lambda_{-1}(\mathcal{R})) \xrightarrow{\cong} \mathcal{K}(M, G; \lambda_{-1}(\nu))$$

Virtual Orbifold Cohomology and Chen-Ruan Cohomology

Let s_g be the Chen-Ruan degree shifting number for a component of $I(T^*\mathcal{X})$ and σ_g the virtual degree shifting number for $I(\mathcal{X})$. Then, it is a fun exercise to show that

$$s_g = \sigma_g.$$

Therefore, the isomorphism of theorem **??** is a graded isomorphism. (For more on gradings we refer the reader to [Hep10]. For related work we refer the reader to [EJK12b, EJK10, EJK12a] where there is alternative approaches to some very related results).

Virtual Orbifold Cohomology and Chen-Ruan Cohomology

For ordinary manifolds, Viterbo [Vit99], Salamon-Weber [SW06] and Abbondandolo-Schwarz [AS06] have constructed isomorphisms between a particular flavor of the Floer homology of the cotangent bundle T^*M and the ordinary homology of the free loop space

$$HF_*(T^*M) \simeq H_*(\mathcal{L}M).$$

Abbondandolo and Schwarz have proved that the pair of pants product in Floer cohomology of the cotangent corresponds to a product in the homology of the loop space, defined via Morse theory, which Antonio Ramirez and Ralph Cohen [CV] proved is the Chas-Sullivan product. One of the main conjectures in the field states that the symplectic field theory on the left-hand side corresponds to the string topology on the right-hand side. Here we should also mention that, for a wide class of manifolds, it has been shown that Floer cohomology is isomorphic to Quantum cohomology [PSS96].

These results are in line with those conjectures.

Virtual Orbifold Cohomology and Chen-Ruan Cohomology

It is routinary to generalize these results to non-global orbifolds.

A. Adem, J. Leida, and Y. Ruan, *Orbifolds and stringy topology*, vol. 171, Cambridge University Press, 2007.

A. Abbondandolo and M. Schwarz, *On the floer homology of cotangent bundles*, Communications on pure and applied mathematics **59** (2006), no. 2, 254–316.

A. Blumberg, R. Cohen, and C. Teleman, *Open-closed field theories, string topology, and hochschild homology*, Alpine Perspectives on Algebraic Topology: Third Arolla Conference on Algebraic Topology, August 18-24, 2008, Arolla, Switzerland, vol. 504, Amer Mathematical Society, 2009, p. 53.

R. Cohen and V. Godin, *A polarized view of string topology*, Topology, geometry and quantum field theory. Proceedings of the 2002 Oxford symposium in honour of the 60th birthday of Graeme Segal (Ulrike Tillmann, ed.), London Mathematical Society Lecture Note Series, no. 308, Cambridge University Press, 2004, arXiv:math.AT/0303003, pp. 127–154.

Weimin Chen and Yongbin Ruan, *A new cohomology theory of orbifold*, Comm. Math. Phys. **248** (2004), no. 1, 1–31. MR MR2104605 (2005j:57036)

M. Chas and D. Sullivan, *String topology*, arXiv:math.GT/9911159.

R. Cohen and A. Voronov, *Notes on string topology*, arXiv:math.GT/0503625.

D. Edidin, T.J. Jarvis, and T. Kimura, *Logarithmic trace and orbifold products*, Duke Mathematical Journal **153** (2010), no. 3, 427–473.

______, *New products, chern classes, and power operations in orbifold k-theory*, arXiv preprint arXiv:1202.0603 (2012).

______, *A plethora of inertial products*, arXiv preprint arXiv:1209.2068 (2012).

- B. Fantechi and L. Göttsche, *Orbifold cohomology for global quotients*, Duke Math. J. **117** (2003), no. 2, 197–227. MR 1 971 293
- R. Hepworth, *The age grading and the chen–ruan cup product*, Bulletin of the London Mathematical Society **42** (2010), no. 5, 868–878.
- Ralph M. Kaufmann, *Orbifolding frobenius algebras*, International Journal of Mathematics **14, Issue: 6** (2003), 573–617.
- E. Lupercio, B. Uribe, and M. Xicoténcatl, *The loop orbifold of the symmetric product*, Journal of Pure and Applied Algebra **211** (2007), no. 2, 293–306.
- S. Piunikhin, D. Salamon, and M. Schwarz, *Symplectic Floer-Donaldson theory and quantum cohomology*, Contact and symplectic geometry (Cambridge, 1994), Publ. Newton Inst., vol. 8, Cambridge Univ. Press, Cambridge, 1996, pp. 171–200. MR MR1432464 (97m:57053)

D. Quillen, *Elementary proofs of some results of cobordism theory using steenrod operations*, Advances in Math **7** (1971), no. 29-56, 1971.

D.A. Salamon and J. Weber, *Floer homology and the heat flow*, Geometric And Functional Analysis **16** (2006), no. 5, 1050–1138.

C. Viterbo, *Functors and computations in Floer homology with applications. I*, Geom. Funct. Anal. **9** (1999), no. 5, 985–1033. MR MR1726235 (2000j:53115)