From seen to unseen Lagrangians via algebraic geometry

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AMS Special session: Advances in symplectic geometry and topology
Mathematical Congress of the Americas, Guanajuato, Mexico

August 6, 2013

Calabi–Yau (CY) mirror symmetry

- A *CY variety* will mean a smooth, proper algebraic variety X over a field, together with a volume form Ω, that is, a trivialization of the canonical line bundle $\mathcal{K}_X$.
- Roughly (and incorrectly), mirror symmetry is an involution

$$(X, \Omega; \omega + ib) \longleftrightarrow (\check{X}, \check{\Omega}; \check{\omega} + i\check{b})$$

  on CY varieties (X, Ω) over $\mathbb{C}$ equipped with complexified Kähler forms $\omega + ib$ on X^{an}.
- Deformations of the class $[\omega + ib] \in H^{1,1}(X)$ correspond to first-order deformations of the variety $\check{X}$ in $H^1(T\check{X})$.
- More accurately (if incompletely), mirror symmetry is an involution on *maximally degenerating 1-parameter families of projective CY varieties.*
  Complexified Kähler deformations of a fiber correspond to complex deformations of a fiber in the mirror family.

Example of a projective 1-parameter degeneration

- We have to view the 1-parameter degenerations both algebraically and analytically. For example, the *Tate curve* $\underline{\mathrm{Tate}}$ is a projective algebraic curve

$$\{y^2 + xy = x^3 + a_4(q)x + a_6(q)\} \subset \mathbb{P}^2(\mathbb{Z}[[q]])$$

with a volume form $\Omega = dx/(2y + x)$.

- The series a_4, $a_6 \in \mathbb{Z}[[q]]$ have radius of convergence 1, so, by viewing them as holomorphic functions in a variable q in the unit disc $\Delta \subset \mathbb{C}$, we can make an analytic subvariety

$$\underline{\mathrm{Tate}}^{an} \subset \mathbb{CP}^2 \times \Delta.$$

The fibers T_q of $\underline{\mathrm{Tate}}^{an} \to \Delta$ for $q \neq 0$ are smooth of genus 1.

- T_q carries a Kähler form obtained by pulling back the Fubini–Study form on $\mathbb{CP}^2$, so it's a symplectic manifold.
- $\underline{\mathrm{Tate}}$ is mirror to itself.

Projective 1-parameter CY degenerations

- A formulation of mirror pairs could identify a class of proper schemes
$$\mathcal{X} \to \operatorname{Spec} \mathbb{Q}[\![q]\!],$$
smooth after inverting q, equipped with ample line bundles $\mathcal{L}$ and trivializations Ω of the dualizing sheaf; and an involution
$$(\mathcal{X} \to \operatorname{Spec} \mathbb{Q}[\![q]\!], \mathcal{L}, \Omega) \longleftrightarrow (\check{\mathcal{X}} \to \operatorname{Spec} \mathbb{Q}[\![q]\!], \check{\mathcal{L}}, \check{\Omega}).$$
- $\mathcal{X}$ and $\check{\mathcal{X}}$ should be defined as projective schemes over the ring of complex power series with positive radius of convergence, so that one can make analytic models
$$\mathbb{X} \subset \mathbb{C}P^N \times \Delta(r), \quad \check{\mathbb{X}} \subset \mathbb{C}P^N \times \Delta(r).$$
- Gross–Siebert's *toric degenerations program* provides algebraic mirror pairs, but convergence of the defining series is currently missing.

Formulations of mirror symmetry

Here are names of some formulations of CY mirror symmetry.

$$
\begin{tikzcd}
\textit{toric-degenerative} \arrow[r, "?"] & \textit{homological} \arrow[r, "?"] \arrow[dr, "\checkmark\ \text{new!}"] & \textit{Hodge-theoretic} \arrow[r, "\checkmark"] & \textit{enumerative} \\
& & \textit{Frobenius-algebraic} \arrow[ur, "\checkmark\ \text{easy}"'] &
\end{tikzcd}
$$

The arrows are implications, conjectured or proven.

From partial HMS to enumerative mirror symmetry

Today, the standing assumption will be a *partial* statement of homological mirror symmetry (HMS)—it will not involve arbitrary coherent complexes, but only line bundles.

$$\begin{tikzcd}
\textit{toric-degenerative} \arrow[r, "?"] & \textit{Partial HMS} \arrow[r, "\checkmark", "\text{new!}"'] & \textit{HMS} \arrow[r, "?"] \arrow[d, "\checkmark"', "\text{new!}"] & \textit{Hodge-theoretic} \arrow[r, "\checkmark"] & \textit{enumerative} \\
 & & \textit{Frobenius-algebraic} \arrow[urr, "\checkmark"'] & &
\end{tikzcd}$$

I'll report that 'partial HMS' implies

1. *full* HMS;
2. *Frobenius-algebraic* mirror symmetry: an isomorphism of Frobenius algebras between quantum cohomology and tangential cohomology of the mirror; and therefore
3. *enumerative* mirror symmetry: a generating series that counts rational curves equals a Yukawa coupling on the mirror.

Homological mirror symmetry (HMS)

- HMS involves a version of the Fukaya A_∞-category $\mathcal{F} = \mathcal{F}(X)$ of Lagrangian submanifolds $L \subset X$.
- I'll be vague about the version I have in mind, because the results rely only on general properties. Among them, $\mathcal{F}$ should be a split-closed triangulated A_∞ category, defined over a field $\mathbb{K}$ which contains $\mathbb{Q}[\![q]\!][q^{-1}]$, with a weak CY structure $\mathcal{F} \simeq \mathcal{F}^\vee[-n]$.
- On the other side of the mirror we have a CY variety $\check{\mathcal{X}}_{\mathbb{K}} \to \operatorname{Spec} \mathbb{K}$, and (a DG model for) its derived category $D\check{\mathcal{X}}_{\mathbb{K}}$ of bounded complexes of coherent sheaves.
- **HMS** asks for an A_∞ quasi-equivalence $\mathcal{F} \simeq D\check{\mathcal{X}}_{\mathbb{K}}$.
- **Partial HMS** involves the full subcategory $\mathcal{B} \subset D\check{\mathcal{X}}_{\mathbb{K}}$ with objects $\{\check{\mathcal{L}}^{\otimes r}\}_{r \geq 0}$. It asks for a fully faithful A_∞ embedding $\mathcal{B} \hookrightarrow \mathcal{F}$.

  More generally, one may take $\mathcal{B}$ to be any split-generating full subcategory of $D\check{\mathcal{X}}_{\mathbb{K}}$.

Hypothesis: partial HMS with maximally unipotent monodromy

We suppose:

- Partial HMS holds: we have a fully faithful functor $\mathcal{B} \to \mathcal{F}$, where $\mathcal{B} \subset D(\check{\mathcal{X}})$ is formed from powers of the polarizing line bundle.
- We have an analytic model

$$\check{\mathbb{X}} \subset \mathbb{C}P^N \times \Delta^*(r)$$

for the mirror family. Write $\check{X}_q$ for the fiber over $q \in \Delta^*(r)$.
- The monodromy $T \in \operatorname{Aut} H^n(\check{X}_q; \mathbb{C})$ (where $q \neq 0$) is *maximally unipotent*, i.e.

$$(T - I)^{n+1} = 0, \quad (T - I)^n \neq 0 \quad (n = \dim_{\mathbb{C}} \check{X}_q).$$

Up to a base-change $q \mapsto q^k$, it's automatic that $(T - 1)^{n+1} = 0$. Frobenius-algebraic (or Hodge-theoretic) mirror symmetry implies $(T - 1)^n \neq 0$, so in that sense, our monodromy assumption does not reduce generality.

From seen to unseen Lagrangians

Theorem (P.–Sheridan.)

Assuming partial HMS with maximally unipotent monodromy, any chosen embedding $\mathcal{B} \to \mathcal{F}$ extends—uniquely, up to natural quasi-isomorphism—to a $\mathbb{K}$-linear quasi-equivalence $D\check{\mathcal{X}}_{\mathbb{K}} \simeq \mathcal{F}$.

- So to prove HMS, it's enough to prove it for some collection of Lagrangians which in practice you can 'see'—the mirrors to the line bundles $\check{\mathcal{L}}^{\otimes r}$.
- X might be teeming with other Lagrangians $L \subset X$ that you don't see. But algebraically, they are built from the seen ones.
- The proof is short but invokes powerful tools:
  1. Abouzaid's generation criterion for Fukaya categories;
  2. the Hochschild structure of algebraic varieties;
  3. the limiting mixed Hodge structure on the cohomology of a degenerating complex algebraic variety.

A ring isomorphism

Theorem (P.–Sheridan.)

Assuming partial HMS with maximally unipotent monodromy, one has a canonical isomorphism of graded unital $\mathbb{K}$-algebras

$$\kappa \colon QH^*(X) \to HT^*(\check{\mathcal{X}}_{\mathbb{K}})$$

from (small) quantum cohomology $QH^*(X) = H^*(X; \mathbb{K})$ to the tangential cohomology

$$HT^*(\check{\mathcal{X}}_{\mathbb{K}}) = \bigoplus_{p+q=*} H^p(\Lambda^q \mathcal{T}\check{\mathcal{X}}_{\mathbb{K}})$$

which maps the symplectic class $[\omega] \in QH^2(X)$ to the Kodaira–Spencer class $\theta = KS(q(d/dq)) \in H^1(\mathcal{T})$.

Meaning of θ: The family $\check{\mathbb{X}} \to \Delta^*$ is a map $\gamma \colon \Delta^* \to \mathcal{M}$ into CY moduli space satisfying the ODE $q(d\gamma/dq) = \theta \circ \gamma$.

Frobenius-algebraic mirror symmetry

Theorem (P.–Sheridan)

We have already stated that partial HMS with maximally unipotent monodromy implies that one has an isomorphism of graded unital $\mathbb{K}$-algebras

$$\kappa\colon QH^*(X) \to HT^*(\check{\mathfrak{X}}_{\mathbb{K}}), \quad \kappa[\omega] = \theta.$$

In addition, κ is a map of Frobenius algebras: for $c \in QH^{2n}(X)$, one has

$$\int_X c = \int_{\check{X}_q} \check{\Omega}_q \wedge (\kappa(c) \cdot \check{\Omega}_q) \in \mathbb{K}.$$

$\check{\Omega}_q$ is the restriction to $\check{X}_q$ of the unique relative volume $\check{\Omega}$ on $\check{\mathfrak{X}}_{\mathbb{K}} \to \operatorname{Spec} \mathbb{K}$ for which Floer–Poincaré duality corresponds under HMS to Serre duality:

$$HF(L_0, L_1) \cong HF(L_1, L_0)^\vee \quad \longleftrightarrow \quad \operatorname{Ext}(\mathcal{E}_0, \mathcal{E}_1) \cong \operatorname{Ext}(\mathcal{E}_1, \mathcal{E}_0)^\vee.$$

Enumerative mirror symmetry

The following enumerative mirror symmetry statement follows from Frobenius-algebraic mirror symmetry ($\check{\Omega}$ as before):

Corollary

Partial HMS with maximally unipotent monodromy implies that

$$\int_X [\omega]^{\star n} = \int_{\check{X}_q} \check{\Omega}_q \wedge \left(q \frac{d}{dq} \right)^n \check{\Omega}_q \in \mathbb{Q}[\![q]\!] .$$

We also have an exchange of summed Hodge numbers:

Corollary

Partial HMS with maximally unipotent monodromy implies that

$$b_k(X) = \sum_{i+j=k} \dim H^j(\Lambda^i \mathfrak{T}\check{\mathfrak{X}}_{\mathbb{K}}) = \sum_{i+j=k} h_{n-i,j}(\check{X}).$$

Where are we?

$$
\begin{tikzcd}
\textit{toric-degenerative} \arrow[r, "?"] & \textit{Partial HMS} \arrow[r, "\checkmark", "\text{new!}"'] & \textit{HMS} \arrow[r, "?"] \arrow[d, "\checkmark"', "\text{new!}"] & \textit{Hodge-theoretic} \arrow[r, "\checkmark"] & \textit{enumerative} \\
 & & \textit{Frobenius-algebraic} \arrow[urr, "\checkmark"'] & &
\end{tikzcd}
$$

- We're missing a proof that HMS implies Hodge-theoretic mirror symmetry: an isomorphism of $QH^*(X)$ with the algebraic de Rham cohomology of $H^*_{DR}(\check{\mathfrak{X}}_{\mathbb{K}})$ which respects *variations of Hodge structure*.
- We're missing a proof that toric-degenerative mirror pairs obey partial HMS. However, Gross and Siebert have already presented the glimmerings of an argument.