Equivariant Extensions of Differential Forms on non-compact Lie groups

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Let M be a manifold, G a Lie group, g its lie algebra and consider an action of G on M.

Cartan model for equivariant cohomology:

$$\Omega_G(M) = ((\Omega^* M \otimes S(g^*))^g, d + \iota)$$

Elements can be seen as $f : g \to \Omega^* M$, polinomial in g, with differentials

$$(df)X := df(X) \quad (\iota f)X := \iota_X f(X).$$

Theorem. [Cartan] For G compact Lie group

$$H^*(\Omega_G(M)) \cong H^*(EG \times_G M; \mathbf{R})$$

For G non compact, Getzler associated to the differential forms of the nerve of the groupoid action the complex

$$C^k(G, S(g^*) \otimes \Omega^* M) = \{f : G^k \times g \to \Omega^* M\}$$

with f polynomial on g, and $f = 0$ if any entry in G^k is the identity. And with differentials

$$(df)(g_1, ..., g_k|X) := (-1)^k df(g_1, ..., g_k|X)$$

$$(\iota f)(g_1, ..., g_k|X) := (-1)^k \iota_X f(g_1, ..., g_k|X)$$

$$\overline{d}(g_0, ..., g_k|X) := f(g_1, ..., g_k|X) + \sum_{i=1}^{k} (-1)^k f(g_0, ..., g_{i-1}g_i, ..., g_k|X)$$

$$+(-1)^{k+1} g_k f(g_0, ..., g_{k-1}|Ad(g_k^{-1})X)$$

$$\bar{\iota} f(g_1, ..., g_k|X) = \sum_{i=0}^{k} \frac{\partial}{\partial t} f(g_1, .., g_{i-1}, e^{tX_i}, g_{i+1}, ..., g_k|X))$$

$$X_i := Ad(g_{i+1} \cdots g_k)X.$$

$$d_G := d + \iota + \bar{d} + \bar{\iota}, \qquad (d_G)^2 = 0$$

$$H^*(G, S(g^*) \otimes \Omega^* M) := H^*(C^*(G, S(g^*) \otimes \Omega^* M), d_G)$$

Theorem [Getzler]

$$H^*(EG \times_G M, \mathbf{R}) \cong H^*(G, S(g^*) \otimes \Omega^* M)$$

Filtering $F_p = \bigoplus_{deg(f)>p} C^*(G, S(g^*) \otimes \Omega^* M)$, we obtain

$$E_0 = C^*(G, S(g^*) \otimes \Omega^* M)$$

$$E_1 = H^*(C^*(G, S(g^*) \otimes \Omega^* M), \bar{d})$$

$$E_1^{*,0} = (S(g^*) \otimes \Omega^* M)^G = \Omega_G M, \quad d_1 : E_1^{*,0} \to E_1^{*+1,0} \qquad d_1 = d + \iota$$

$$E_2^{*,0} = H^*(\Omega_G M, d + \iota)$$

$$E_2^{*,2} \to E_\infty^{*,0} \subset H^*(G, S(g^*) \otimes \Omega^* M)$$

Proposition. There is a canonical homomorphism

$$H^*(\Omega_G M, d+\iota) \longrightarrow H^*(G, S(g^*) \otimes \Omega^* M).$$

Proposition. Consider the projection map

$$H^*(\Omega_G M, d+\iota) \longrightarrow H^*(M),$$

and let ω be a closed form on M. There exists an extension $\widetilde{\omega}$ of ω in the Cartan complex if and only if the cohomology class $[\omega]$ can be extended to an equivariant cohomology class, and this class lies in $E_\infty^{*,0}$.

Example: $G = SL(n, \mathbf{R})$ and $M = pt$.

$E_1 = H^*(C^*(G, S(g^*)), \overline{d})$ by Van-Est and $k = so(n)$

$= H^*(g, k; S(g^*))$ because g is reductive

$= H^*(g, k; \mathbf{R}) \otimes S(g^*)^g$ with

$H^*(g, k; \mathbf{R}) = H^*(SU(n)/SO(n)) = \Lambda[h_3, h_5, ..., h_{\langle n \rangle}]$ for $|h_i| = 2i-1$ and $\langle n \rangle$ the largest odd number $\leq n$, and

$S(g^*)^g = \mathbf{R}[c_2, c_3, ..., c_n]$ with $|c_i| = 2i$.

$H^*(G, S(g^*)) = H^*(BSL(n, \mathbf{R})) = H^*(BSO(n)) = \mathbf{R}[c_2, c_4, ..., c_{[n/2]}]$
and in particular:

$S^2(g^*)^g = H^4(G, S(g^*))$ impliying $E_2^{4,0} = E_\infty^{4,0} = H^4(G, S(g^*))$.

Gauged WZW terms (2d bosonic σ-model)

Γ a simple, connected and simply connected matrix Lie grouup.

Σ Riemann surface without boundary.

Bosonic fields $g : \Sigma \to \Gamma$.

Bosonic WZW functional:

$$I(g) = KT(g) + i \int_B \widetilde{g}^* \omega$$

With $\partial B = \Sigma$, $\widetilde{g} : B \to \Gamma$, $\widetilde{g}|_\Sigma = g$

and

$$\omega = \frac{1}{2\pi} \mathrm{Tr}((g^{-1} dg)^3).$$

$I(g)$ is invariant under $\Gamma_L \times \Gamma_R$, where

$$(\Gamma_L \times \Gamma_R) \times \Gamma \to \Gamma, \quad ((a,b),c) \mapsto acb^{-1}$$

and one could try to gauge it.

For $G \subset \Gamma_L \times \Gamma_R$, generalize the fields to sections $g : \Sigma \to X$ with

$$\Gamma \to X \to \Sigma$$

a bundle with fiber Γ and structural group G.

For A a connection on X, this means to find a functional $I(g, A)$, gauge invariant, which reduces to $I(g)$ when $A = 0$.

Denote

$$g \to Lie(\Gamma) \times Lie(\Gamma), \quad a \mapsto (T_{a,L}, T_{b,R})$$

and $X_a \in VectorFields(\Gamma)$ with $(X_a)_g = T_{a,L}g - gT_{b,R}$.

Physicist noted that the gauge extension exists if anf only if for all $a, b \in g$

$$\mathrm{Tr}(T_{a,L}T_{b,L} - T_{a,R}T_{b,R}) = 0.$$

The subgroups G that satisfy all these equations were called **anomally free** groups.

Witten noted that for

$$\lambda_a = \frac{1}{4\pi}\mathsf{Tr}(T_{a,L}(dg\ g^{-1}) + T_{a,R}(g^{-1}dg))$$

one gets an invariant differential form in the Cartan complex

$$\widetilde{\omega} = \omega - \lambda_a \Omega_a \in (S(g^*) \otimes \Omega^* \Gamma)^G$$

which moreover satisfy

$$\iota_a \omega = d\lambda_a$$

and

$$(d + \Omega^a \iota_a)\widetilde{\omega} = \Omega^a \Omega^b \iota_a \lambda_b \in S^2(g^*)^g$$

where the coefficient of $\Omega^a \Omega^b$ is

$$\iota_a \lambda_b + \iota_b \lambda_a = \frac{1}{2\pi}\mathsf{Tr}(T_{a,L}T_{b,L} - T_{a,R}T_{b,R})$$

Theorem [Witten] The group G is anomally free if and only if

$$(d + \Omega^a \iota_a)\widetilde{\omega} = 0.$$

Hence if G is compact Lie, the group is anomally free if and only if $[\omega]$ can be extended to $H^3(\Gamma \times_G EG, \mathbf{R})$.

Theorem [GPU] The group $G = SL(n, \mathbf{R})$ is anomally free if and only if $[\omega]$ can be extended to $H^3(\Gamma \times_G EG, \mathbf{R})$.

Proof:

obstruction $d_4[\omega] \in H^4(BG) = S(g^*)^g$, and therefore $d_4[\omega] = (d + \iota)\widetilde{\omega}$.

For example we have cancellation of anomallies for $SL(n, \mathbf{R})$ whenever the action is diagonal.

There is no cancellation of anomallies when $SL(n, \mathbf{R}) \subset \Gamma_L$ for $n \geq 3$.

THANKS