# Transversally-Hamiltonian Group Actions Around Symplectic Leaves

Yury Vorobiev

University of Sonora
(Hermosillo, México)

Mathematical Congress of the Americas 2013,
Guanajuato, August 5-9, 2013

## Contents

Goal: To describe an averaging procedure for Poisson structures with singular symplectic foliations.

## Poisson Manifold

Let $(M, \Pi)$ be a Poisson manifold with Poisson tensor $\Pi \in \Gamma(\bigwedge^2 TM)$,

$$[\Pi, \Pi]_{\text{Sch}} = 0$$

$$\Downarrow$$

- the characteristic distribution

$$C := \Pi^\sharp(T^*M) \equiv \text{Span}\{X_f = \Pi^\sharp df \mid f \in C^\infty_M\}$$

- the symplectic foliation $(\mathcal{S},\omega)$ with (smooth) leafwise symplectic form $\omega$

$$T\mathcal{S} = C$$

$$\omega(X_f, X_g) = \Pi(df, dg)$$

- the Dirac structure $D \subset TM \oplus T^*M$,

$$D = \text{graph}(\Pi) = \{(\Pi^\sharp(\alpha), \alpha) \mid \alpha \in T^*M\}$$

# Leaf Preserving Noncanonical Actions

- Let $G$ be a connected compact Lie group and $\mathfrak{g}$ its Lie algebra
- Suppose we are given a smooth action $\Phi : G \times M \to M$ such that $\Phi$ is leaf preserving on $(M, \mathcal{S})$:

$$a_M(m) = \frac{d}{dt}\bigg|_{t=0} \Big[\Phi_{\exp(ta)}(m)\Big] \in T_m\mathcal{S} \qquad \forall\, a \in \mathfrak{g},$$

- but $\Phi$ is not canonical (Poisson) on $(M, \Pi)$

**Question:** Can we recover a $G$-invariant Poisson tensor $\overline{\Pi}$ (or Dirac structure $\overline{D}$) from $(M, \Pi, G, \Phi)$?

**Remark**

*The $G$-average of $\Pi$ is not Poisson, in general.*

# Averaging the Leafwise Symplectic Form

- Consider the $G$-average of the leafwise symplectic form $\omega$

$$\langle \omega \rangle^G := \int_G \Phi_g^* \, \omega \; dg$$

- $\langle \omega \rangle^G$ is a $G$-invariant leafwise 2-form, closed on each leaf of $\mathcal{S}$

# Smoothness and Nondegeneracy Conditions

- The smoothness of $\langle\omega\rangle^G$: the associated distribution $\overline{D} \subset TM \oplus T^*M$,

$$\overline{D}_m : \left\{ (X, \alpha) \in T_m M \oplus T_m^* M \;\middle|\; X \in C_m,\ \alpha|_{C_m} = \mathbf{i}_X \langle\omega\rangle^G \right\}$$

  is a smooth subbundle.

$$\Downarrow$$

$(\mathcal{S}, \langle\omega\rangle^G\ )$ is a smooth presymplectic foliation

$$\Downarrow$$

$G$-invariant Dirac structure $\overline{D}$

- The nondegeneracy condition: $\langle\omega\rangle^G$ is symplectic on each leaf of $\mathcal{S}$

$$\Downarrow$$

$G$-invariant Poisson structure $\overline{\Pi}$

## Criterion

### Proposition

*If there exists an* $\mathbb{R}$*-linear mapping*

$$\rho : \mathfrak{g} \to \Omega^1(M),$$

*such that*

$$\mathbf{i}_{a_M}\omega = (\text{the pull-back of } \rho_a) \qquad \forall\, a \in \mathfrak{g},$$

*then the smoothness condition holds and the* $G$*-invariant Dirac structure* $\overline{D}$ *is related with the original one* $D$ *by an exact gauge transformation, Bursztyn)*

$$\overline{D} = \{(X, \alpha - \mathbf{i}_X dQ) \mid (X, \alpha) \in D\},$$

*for a certain* $Q \in \Omega^1(M)$*.*

# Main Observation (Poisson case)

One can apply the criterion to a neighborhood of a (singular) symplectic leaf $S$ of $(M, \Pi)$

## Claim

*If the compact Lie group $G$ acts in a Hamiltonian way with respect to the transverse Poisson structure of the symplectic leaf $S$, then $\Pi$ admits a $G$-invariant realization: In a neighborhood of $S$, there exists a $G$-invariant Poisson tensor $\overline{\Pi}$ which is isomorphic to $\Pi$,*

$$\phi^* \Pi = \overline{\Pi}.$$

**Application**: Normal forms of adiabatic type for Hamiltonian dynamics around a symplectic leaf.

**Tools**:

- Coupling Poisson structures (Vorobiev, Vaisman).
- The averaging procedure for Poisson connections (Marsden, Montgomery & Ratiu).

## $\mathcal{F}$-Coupling Poisson Structures

Let $\mathcal{F}$ be a regular foliation on a manifold $M$

- $\mathbb{V} := T\mathcal{F}$ the *tangent bundle*
- $\mathbb{V}^0 \subset T^*M$ the *annihilator* of $\mathbb{V}$

### Definition

*A Poisson bivector field* $\Pi \in \Gamma(\bigwedge^2 TM)$ *is* $\mathcal{F}$*-coupling if the distribution*

$$\mathbb{H} := \Pi^\sharp(\mathbb{V}^0)$$

*is a normal bundle of* $\mathcal{F}$*,*

$$TM = \mathbb{H} \oplus \mathbb{V}.$$

# Splitting

## Proposition

*Every $\mathcal{F}$-coupling Poison tensor* $\Pi$ *has the decomposition*

$$\Pi = \Pi_{\mathbb{H}} + \Pi_{\mathbb{V}},$$

*where*

- *the "regular part"* $\Pi_{\mathbb{H}} \in \Gamma(\bigwedge^2 \mathbb{H})$ *is a bivector field of constant rank,*

$$\operatorname{rank}(\Pi_{\mathbb{H}}) = \dim \mathbb{H} = \operatorname{codim}\mathcal{F}.$$

- *the "singular part"* $\Pi_{\mathbb{V}} \in \Gamma(\bigwedge^2 \mathbb{V})$ *is a leaf-tangent Poisson tensor,*

$$\Pi_{\mathbb{V}}^{\sharp}(T^*M) \subset T\mathcal{F}.$$

# Properties

- the characteristic distribution of $\Pi$

$$\Pi^\sharp(T^*M) = \mathbb{H} \oplus \Pi^\sharp_{\mathbb{V}}(\mathbb{H}^0).$$

- the symplectic leaves of $\Pi$ intersect the leaves of $\mathcal{F}$ transversally and symplectically.

- the set of singular points

$$\mathrm{Sing}(\Pi) = \mathrm{Sing}(\Pi_{\mathbb{V}}).$$

- $\Pi_{\mathbb{H}}$ is Poisson iff $\mathbb{H}$ is integrable.

## Basic Example

A natural class of coupling Poisson structures comes from a neighborhood of a symplectic leaf.

### Proposition

*Let $\Pi$ be a Poisson tensor on $(M, \mathcal{F})$ and $S \subset M$ a symplectic leaf such that*

$$T_S M = TS \oplus T_S \mathcal{F}.$$

*Then, in a tubular neighborhood $N$ of $S$ in $M$, $\Pi$ is an $\mathcal{F}$-coupling Poisson tensor,*

$$\Pi\big|_N = \Pi_{\mathbb{H}} + \Pi_{\mathbb{V}},$$

*with*

$$\mathbb{H}\big|_S = TS, \qquad \Pi_{\mathbb{V}} = 0 \quad \text{on} \quad S.$$

### Definition

*The leaf-tangent Poisson bivector field $\Pi_V$ on $N$ is said to be the $\mathcal{F}$-transverse Poisson structure of $\Pi$ around the symplectic leaf $S$.*

**Note**: $S$ is regular $\Leftrightarrow \Pi_{\mathbb{V}} \equiv 0$.

# Main Results

## Transversally-Hamiltonian $G$-actions

On a foliated manifold $(M, \mathcal{F})$ we are given

- a leaf preserving action $\Phi : G \times M \to M$ of a connected compact Lie group $G$.
- a Poisson tensor $\Pi$ with an embedded symplectic leaf $S \subset M$ such that

$$T_S M = TS \oplus T_S \mathcal{F}.$$

Assume that

- the $G$-action is canonical relative to the $\mathcal{F}$-transverse Poisson structure $\Pi_{\mathbb{V}}$ of $S$, with momentum map $\mathbb{J} : N \to \mathfrak{g}^*$

$$a_M = \Pi_{\mathbb{V}}^{\sharp} \, d\mathbb{J}_a, \qquad \forall \, a \in \mathfrak{g}.$$

# Averaging Theorem

## Theorem

*In a tubular neighborhood $\overline{N}$ of the leaf $S$, there exists an $\mathcal{F}$-coupling Poisson structure $\overline{\Pi} = \overline{\Pi}_{\mathbb{H}} + \Pi_{\mathbb{V}}$ which is G-invariant and possesses the properties:*

- *the characteristic distributions of $\overline{\Pi}$ and $\Pi$ coincide on $\overline{N}$.*
- *the leafwise symplectic form of $\overline{\Pi}$ is just $\langle\omega\rangle^G$.*
- *Poisson structures $\overline{\Pi}$ and $\Pi$ are related by*

  $$\overline{\Pi}^\sharp = \Pi^\sharp \circ (\mathrm{Id} - (dQ)^\sharp \circ \Pi^\sharp)^{-1},$$

  *for a certain $Q \in \Gamma(\mathbb{V}^0_{\overline{N}})$*

- *the germs at $S$ of $\overline{\Pi}$ and $\Pi$ are isomorphic,*

  $$\phi^*\Pi = \overline{\Pi}, \qquad \phi\,|_S = \mathrm{id}.$$

# Averaging Theorem

**Remark**

*The regularizing mapping* $\rho : \mathfrak{g} \to \Omega^1(M)$ *is given by*

$$\rho_a = -d_{\mathcal{F}} \mathbb{J}_a, \qquad (a \in \mathfrak{g})$$

**Remark**

*The gauge 1-form* $Q$ *can be expressed in terms of the momentum map* $\mathbb{J}$*.*

In the case $G = \mathbb{S}^1 = \mathbb{R} \diagup 2\pi\mathbb{Z}$:

$$Q = \frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(\mathrm{Fl}^t_{\Pi^\sharp_{\mathbb{V}} d\mathbb{J}})^* \, d_{\mathbb{H}} \, \mathbb{J} \, dt$$

where $d_{\mathbb{H}} = \mathrm{pr}_H \circ d$ is the $\mathbb{H}$-dependent covariant exterior derivative

# Adiabatic Condition

## Proposition

*The following two statements are equivalent:*

- *the $G$-action is Hamiltonian relative to the "averaged" Poisson structure* $\overline{\Pi}$
- *one can choose the momentum map* $\mathbb{J} : \overline{N} \to \mathfrak{g}^*$ *associated to the transversally-Hamiltonian $G$-action so that*

$$\langle d_{\mathbb{H}} \mathbb{J} \rangle^G = 0.$$

## Remark

*The adiabatic condition (Marsden,Montgomery & Ratiu) can be expressed in cohomological terms.*

## Infinitesimal Version

Let $E = T_S M/TS$ be the normal bundle of the leaf $S$ equipped with the fiberwise linear $G$-action:

### Corollary

*There exists a equivariant exponential map* $\mathbf{f} : N_S(E) \to N_S(M)$ *such that the first order approximation of the pull-back Poisson structure* $\mathbf{f}^*\overline{\Pi}$ *defined around the zero section* $S \hookrightarrow E$ *gives a* $G$*-invariant realization of the linearized Poisson structure of the symplectic leaf* $S \subset M$ *of* $\Pi$*.*

# Example. Families of Hamiltonian Actions

Let

- $S$ be a connected symplectic manifold (viewed as a parameter space),
- $F$ a Poisson manifold equipped with a family of Hamiltonian actions of a compact Lie group $G$,
- $x_0 \in F^G$ a fixed point of zero rank,

Then, the product Poisson structure $\Pi$ on $M = S \times F$ admits a $G$-invariant realization around the slice $S \times \{x_0\}$.

# Integrable Geometric Data

## Definition

*Geometric data* $(\mathbb{H}, \sigma, P)$ *on a foliated manifold* $(M, \mathcal{F})$ *consist of*

- *a normal bundle* $\mathbb{H}$ *of* $\mathcal{F}$.
- *a horizontal (coupling) 2-form* $\sigma \in \Gamma \bigwedge^2(\mathbb{V}^0)$ *on* $M$
- *a leaf-tangent Poisson tensor* $P \in \Gamma(\bigwedge^2 \mathbb{V})$

Integrability of $(\mathbb{H}, \sigma, P)$ = the structure equations

$$L_X P = 0,$$

$$d_{\mathbb{H}} \sigma = 0,$$

$$\mathrm{pr}_{\mathbb{V}}([X, Y]) = -P^{\sharp} d\sigma(X, Y)$$

for any $\mathbb{H}$-tangent projectable vector fields $X, Y \in \Gamma_{\mathrm{pr}}(\mathbb{H})$.

# Parametrization and Factorization

## Proposition

*Every $\mathcal{F}$-coupling Poisson structure $\Pi$ on $(M, \mathcal{F})$ is equivalent with integrable geometric data $(\mathbb{H}, \sigma, P)$ such that*

$$\sigma\big|_{\mathbb{H}} \quad \textit{is nondegenerate}$$

$$\Downarrow$$

*The structure equations for $(\mathbb{H}, \sigma, P)$ give a factorization of the Jacobi identity for $\Pi$.*

# Coupling **Dirac Structures**

Removing the nondegeneracy condition, we get (*Dufour & Wade, Vaisman*)

## Proposition

*There exists a one-to-one correspondence between*

- *integrable geometric data on* $(M, \mathcal{F})$

*and*

- *Dirac structures* $D \subset TM \oplus T^*M$ *satisfying the coupling condition: the tangent distribution* $\mathbb{H} = \mathbb{H}(D, \mathcal{F})$

$$\mathbb{H}_m := \{Z \in T_m M \,\big|\, \exists\, \alpha \in \mathbb{V}^0 \quad \& \quad (Z, \alpha) \in D\}$$

*is a normal bundle of* $\mathcal{F}$.

# Gauge Equivalence

For a given horizontal 1-form $Q \in \Gamma(\mathbb{V}^0)$, we define the $Q$-transform

$$\mathcal{T}_Q : (\mathbb{H}, \sigma, P) \mapsto (\widetilde{\mathbb{H}}, \widetilde{\sigma}, P)$$

by

$$\widetilde{\mathbb{H}} = \big(\mathrm{Id} - P^\sharp \circ (dQ)^\sharp\big)(\mathbb{H})$$

$$\widetilde{\sigma}^\sharp = \sigma^\sharp - (d_{\mathbb{H}} Q)^\sharp + (dQ)^\sharp \circ P^\sharp \circ dQ^\sharp .$$

**Definition**

*Two integrable geometric data are gauge equivalent if they are related by a $Q$-transform.*

**Proposition**

*The $Q$-transform leaves invariant the subset of all integrable geometric data on $(M, \mathcal{F})$.*

# Induced $G$-Actions

Given a leaf preserving action $\Phi : G \times M \to M$ on $(M, \mathcal{F})$

$$\Downarrow$$

the $G$-action on the set of all geometric data on $(M, \mathcal{F})$

$$\mathcal{T}_g : (\mathbb{H}, \sigma, P) \mapsto (d\Phi_{g^{-1}}(\mathbb{H}),\ \Phi_g^* \sigma,\ \Phi_g^* P)$$

## Criterion

*A $\mathcal{F}$-coupling Poisson (Dirac) structure is invariant with respect to the $G$-action on $(M, \mathcal{F})$ iff the associated integrable data*

$$(\mathbb{H}, \sigma, P) \quad \textit{are} \quad G - \textit{invariant}.$$

## Proposition (Key Observation)

*Let $(\mathbb{H}, \sigma, P)$ be integrable data on $(M, \mathcal{F})$. If a connected compact Lie group $G$ acts canonically on $(M, P)$ with momentum map $\mathbb{J} : M \to \mathfrak{g}^*$, then there exists a gauge 1-form $Q \in \Gamma(\mathbb{V}^0)$ such that the "averaged" data $(\overline{\mathbb{H}}, \overline{\sigma}, P)$ given by*

$$\overline{\mathbb{H}} := \mathrm{Span}\{\langle X \rangle^G \mid X \in \Gamma_{\mathrm{pr}}(\mathbb{H})\}$$

$$\overline{\sigma} := \langle \sigma \rangle^G - \langle dQ \cdot P \cdot dQ \rangle^G$$

*are:*

- *$G$-invariant,*
- *$Q$-gauge equivalent to $(\mathbb{H}, \sigma, P)$,*
- *integrable.*

**Corollary**

*If $\overline{\sigma}\big|_{\overline{\mathbb{H}}}$ is nondegenerate, then the $\mathcal{F}$-coupling Poisson tensor $\overline{\Pi}$ associated to the averaged data $(\overline{\mathbb{H}}, \overline{\sigma}, P)$ is $G$-invariant.*

- In general, the triple $(\overline{\mathbb{H}}, \overline{\sigma}, P)$ induces the $G$-invariant Dirac structure

$$\overline{D} = \operatorname{graph}(\overline{\sigma}\big|_{\overline{\mathbb{H}}}) \oplus \operatorname{graph}(P\big|_{\overline{\mathbb{H}}^0}).$$

# Poisson Diffeomorphism $\phi$

- Let $\Pi_0$ and $\Pi_1$ be two Poisson structures on $(M, \mathcal{F})$ admiting a common symplectic leaf $S \subset M$ which is transverse to $\mathcal{F}$,

$$T_S M = TS \oplus T_S \mathcal{F}$$

## Criterion

*If the integrable geometric data associated to the germs of $\Pi_0$ and $\Pi_1$ at $S$ are gauge equivalent, then there exists a local diffeomorphism $\phi$ around $S$ such that*

$$\phi^* \Pi_0 = \Pi_1 \quad \textit{and} \quad \phi\big|_S = \mathrm{id}.$$

# Moser's Homotopy Method

- a path of integrable geometric data $t \mapsto (\mathbb{H}_t, \sigma_t, P)$ joining the geometric data of $\Pi_0$ and $\Pi_1$
- the family of coupling Poison strictures $t \mapsto \Pi_t$ associated to $(\mathbb{H}_t, \sigma_t, P)$ is well-defined in a neighborhood of $S$ for all $t \in [0, 1]$
- the homotopy equation for $Z_t \in \Gamma(\mathbb{H}_t)$

$$L_{Z_t}\Pi_t \; + \; \frac{d\Pi_t}{dt} = 0$$

is reduced to

$$\mathbf{i}_{Z_t}\sigma_t = -Q.$$

($Q$ is a gauge 1-form vanishing at $S$)

$$\Downarrow$$

Poisson diffeomorphism $\phi \quad = \quad$ time-1 flow of $Z_t$

# Related Results

- Equivariant Splitting Theorem (Miranda & Zung)
- Coupling Cartan-Dirac structures (Wade)
- Poisson transversals (Frejlich & Mărcut)