SYMPLECTIC VORTICES WITH FIXED HOLONOMY AT INFINITY

E. GONZALEZ, A. OTT, C. WOODWARD, AND F. ZILTENER

Abstract. Let Σ be a Riemann surface with cylindrical ends, G a compact, connected Lie group and let X denote a compact symplectic manifold with a Hamiltonian G-action. Given a conjugacy class associated to each end, we define a moduli space of symplectic vortices which generalizes the moduli spaces of parabolic bundles introduced by Mehta and Seshadri. Using the moduli spaces we construct gauged Gromov-Witten invariants for convex target.

Contents

Preliminary version.

1. Introduction

Let Σ be a Riemann surface, G a compact Lie group, and X a compact Hamiltonian G-manifold equipped with an invariant compatible almost complex structure J. A *gauged J-holomorphic map* from Σ to X consists of a principal-bundle-with-connection (P, A) and a section u of the associated X-bundle, holomorphic with respect to the almost complex structure on $P(X)$ determined by the connection A. This space has a natural action of the group of gauge transformations of the bundle. The *moduli space of symplectic vortices* $M(\Sigma, X)$ is obtained from the symplectic quotient construction; this generalizes the Atiyah-Bott construction of the moduli space of flat bundles on Σ, which is the case that X is trivial. In the case that Σ has cylindrical ends, one obtains a natural moduli space $M(\Sigma, X, \underline{\mu})$ by fixing the holonomies $\underline{\mu}$ of the connection around the circles

Date: July 8, 2008.

at infinity. In the case that X is trivial, these are the moduli spaces of parabolic bundles considered by Mehta-Seshadri [8].

In this paper we construct a compactified moduli space $\overline{M}(\Sigma, X, \underline{\mu})$ of *polystable symplectic vortices.* These moduli spaces are expected to play a role in various factorization theorems, in particular, the generalization of orbifold quantum cohomology to continuous Lie groups. In the case of the circle group, part of this project is carried out by Mundet-Tian [6]. A polystable symplectic vortex consists of a symplectic vortex on the principal component, and a collection of holomorphic sphere and cylinder bubbles such that each bubble has at least three special points. (In the case $G = S^1$, the compactification is a special case of [6] in which gradient lines do not appear.) The main result is the following. Let $E : \overline{M}(\Sigma, X, \underline{\mu}) \to \mathbb{R}$ denote the energy functional.

Theorem 1.0.1. *For any collection $\underline{\mu}$ of rational holonomies, and any energy bound $C > 0$, the intersection $\overline{M}(\Sigma, X, \underline{\mu}) \cap E^{-1}(C)$ is a compact Hausdorff space.*

Our proof uses a somewhat different approach to compactness than the one described in [6]. Namely, we use a generalization of the *invariant symplectic action* introduced in Gaio-Salamon [4] that we call the *bi-invariant symplectic action.* This connection depends on a loop x in X and the choice of *two* connections a, a_0 on the trivial G-bundle on X; it measures the equivariant symplectic area necessary to contract x to a a_0-covariant constant loop. Much of the paper is taken up by a definition of this notion of action, and a comparison to the relative symplectic action for clean Lagrangian intersections introduced by Pozniak [10].

We thank Constantin Teleman for encouragement.

2. Vortices with fixed holonomy

2.1. **Bundles with fixed holonomy.** In the case of trivial target the theory that we wish to develop reduces to the Mehta-Seshadri theory of parabolic bundles [8]. Let G be a compact, 1-connected Lie group. We denote by T a maximal torus and W the Weyl group. Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{t}_+$ a choice of positive chamber, so that $\alpha_0 \in \mathfrak{t}^*$ is the highest root. The *Weyl alcove* $\mathfrak{A}$ is the subset of $\mathfrak{t}_+$ defined by

$$\mathfrak{A} := \{\xi \in \mathfrak{t}_+, \quad \alpha_0(\xi) \leq 1\}.$$

The sequence of maps $\mathfrak{A} \to T \to G$ given by exponential and inclusion respectively induce isomorphisms of quotient spaces

$$\mathfrak{A} \cong T/W \cong G/\operatorname{Ad}(G).$$

For any $\mu \in \mathfrak{A}$ we denote by $\mathcal{C}_\mu$ the corresponding conjugacy class.

Let Σ be a compact, oriented surface with $n \geq 0$ boundary components, $P \to \Sigma$ be a principal G-bundle. The space $\mathcal{A}(P)$ of connections on P is an affine space modelled on

$\Omega^1(\Sigma, P(\mathfrak{g}))$ the space of one-forms with values in the adjoint bundle $P(\mathfrak{g})$. The choice of an invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ induces a map

$$\Omega^1(\Sigma, P(\mathfrak{g}))^2 \to \Omega^2(\Sigma), \quad (a_1, a_2) \mapsto \langle a_1 \wedge a_2 \rangle_{\mathfrak{g}}$$

by combining the wedge product and metric. This induces on $\mathcal{A}(P)$ the structure of an infinite-dimensional symplectic manifold with symplectic form given by

$$(a_1, a_2) \mapsto \int_\Sigma \langle a_1 \wedge a_2 \rangle_{\mathfrak{g}}. \tag{1}$$

The group of gauge transformations

$$\mathcal{G}(P) = \{a : P \to P, a(pg) = a(p)g \forall g \in G\}$$

acts on $\mathcal{A}(P)$ via pull-back. The infinitesimal action is given by

$$\Omega^0(\Sigma, P(\mathfrak{g})) \to \Omega^1(\Sigma, P(\mathfrak{g})), \quad \xi \mapsto -\mathrm{d}_A \xi.$$

The action preserves the symplectic form (1) and has moment map given by the curvature

$$\mathcal{A}(\Sigma, G) \to \Omega^2(\Sigma, \mathfrak{g}), \quad A \mapsto F_A.$$

In the case that Σ has boundary, the gauge group fits into an exact sequence

$$1 \to \mathcal{G}_\partial(P) \to \mathcal{G}(P) \to \mathcal{G}(\partial P) \to 1$$

where $\mathcal{G}_\partial(\Sigma)$ is the group of gauge transformations that equal to the identity on the boundary. The symplectic quotient

$$M(P, \partial P) = \mathcal{A}_\flat(P)/\mathcal{G}_\partial(P)$$

may be identified with the moduli space of flat connections with framing on the boundary. The residual group $\mathcal{G}(\partial P)$ acts on $M(P, \partial P)$ by changing the framing on the boundary, with moment map given by restriction to the boundary

$$\Phi : M(P, \partial P) \to \mathcal{A}(\partial P).$$

The $\mathcal{G}(\partial P)$-orbits on $\mathcal{A}(\partial P)$ are parametrized by conjugacy classes, via the map given by measuring the holonomy around each boundary component. For any $\underline{\mu} \in \mathfrak{A}^n$, where n is the number of boundary components, we denote by $\mathcal{O}_{\underline{\mu}} \subset \mathcal{A}(\partial P)$ the orbit labelled by $\mu \in \mathfrak{A}^n$. The symplectic quotient

$$M(P, \underline{\mu}) = \Phi^{-1}(\mathcal{O}_{\underline{\mu}})/\mathcal{G}(\partial P),$$

is the *moduli space of flat connections with fixed holonomies* $\exp(\underline{\mu})$ around the boundary. Let

$$M(\Sigma, \underline{\mu}) = \bigcup_P M(P, \underline{\mu})$$

denote the union over topological types of bundles $P \to \Sigma$, the *moduli space of flat bundles with fixed holonomies.*

Lemma 2.1.1. *There exists a homeomorphism*

$$M(\Sigma, \underline{\mu}) \to \{\varphi \in \operatorname{Hom}(\pi_1(\Sigma), G), \quad \varphi(\gamma_i) \in \mathcal{C}_{\mu_i}, i = 1, \dots, n\} /G.$$

For generic $\underline{\mu}$, the space $M(\Sigma, \underline{\mu})$ is a compact orbifold.

Proof. Any flat bundle is determined by its holonomies up to conjugacy; conversely, it is straightforward to construct from a representation a bundle with that holonomy. For the second claim, see [9]. □

The moduli spaces $M(\Sigma, \underline{\mu})$ have a holomorphic description due to Mehta-Seshadri [8], see also [1], [11], as follows. Let $P_{\mathbb{C}} \to \Sigma$ be a holomorphic principal $G_{\mathbb{C}}$-bundle.

Definition 2.1.2. A *quasiparabolic structure* at a point $z \in \Sigma$ is a reduction of structure group of E_z to a parabolic subgroup $P \subset G$, that is, a point in the quotient $\sigma_z \in E_z/P$. A *parabolic structure* is a quasiparabolic structure $\sigma_z \in E_z/P$ together with conjugacy class $\mathcal{C} \subset G$ of the type specified by the parabolic subgroup. That is, the Levi subgroup L of the parabolic P is isomorphic to the centralizer Z_g of any point g in the conjugacy class $\mathcal{C}$. A parabolic bundle is *semistable* if a certain inequality is satisfied for each reduction of E to a maximal parabolic subgroup (at least, if none of the markings μ_j are contained in the opposite wall of the Weyl alcove, see [11]).

On the set of semistable parabolic bundles one defines a *grade equivalence* relation, which equates parabolic bundles if their associated graded bundles are isomorphic. Let $M_{\mathbb{C}}(\Sigma, \underline{\mu})$ denote the moduli space of grade-equivalence classes of semistable parabolic bundles. Narasimhan-Mehta-Seshadri [8] and extensions [3], [1], [11] show that

Theorem 2.1.3. *$M_{\mathbb{C}}(\Sigma, \underline{\mu})$ is a normal projective variety homeomorphic to $M(\Sigma, \underline{\mu})$.*

In the case that each conjugacy class μ_j has finite order, one can describe $M_{\mathbb{C}}(\Sigma, \underline{\mu})$ in terms of equivariant bundles for a finite group action. Let $\tilde{\Sigma} \to \Sigma$ be a totally ramified $\mathbb{Z}_N$-covering, such that the order of each μ_j divides N. Suppose we are given a $\mathbb{Z}_N$-equivariant holomorphic principal $G_{\mathbb{C}}$-bundle $\tilde{E} \to \tilde{\Sigma}$, with the following property: consider a local trivialization near s_j in which the generator of $\mathbb{Z}_N$ acts by g_j, and suppose that $\exp(\mu_j) = g_j$. At each puncture glue in the trivial bundle $D \times G_{\mathbb{C}}$ over the disk via the transition map

$$(z, g) \mapsto (z, \exp(N \ln(z) \mu_j / 2\pi i) g). \tag{2}$$

The group $\mathbb{Z}_N$ acts freely on the resulting completed bundle and the quotient is a holomorphic principal $G_{\mathbb{C}}$-bundle E over $\overline{\Sigma}$. The bundle E has a parabolic reduction at the fiber s_j, given as the image of $(0, P_j)$ in the local trivialization, where P_j is the parabolic corresponding to ξ_j; this is independent of the local trivializations used above. See [8], [11] for proof of the following.

Proposition 2.1.4. *Mapping $\tilde{E}$ to E defines an equivalence of categories between equivariant bundles on $\tilde{\Sigma}$ and parabolic bundles on Σ preserving the semistability condition.*

2.2. **Gauged holomorphic maps.** We denote by j_0 the standard complex structure on $\mathbb{R} \times S^1 \simeq \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ and denote the coordinates in this space by (s, t). Let $(\Sigma, \omega_\Sigma, j)$ be a Riemann surface without boundary with a compatible area form.

Definition 2.2.1. We say that $(\Sigma, \omega_\Sigma, j)$ has admissible cylindrical ends iff there exists a compact subset $K \subseteq \Sigma$, a non-negative integer n and maps $\phi_i : (0, \infty) \times S^1 \to \Sigma$ that are diffeomorphisms onto their image, for $i = 1, \ldots, n$, such that the following holds. The surface Σ is the disjoint union of K and the images of the maps ϕ_i. Furthermore, $\phi_i^* j = j_0$ and there exist numbers $\kappa_i, C_i > 0$ such that

$$\phi_i^* \omega_\Sigma = C_i^2 e^{-2\kappa_i s} ds \wedge dt.$$

Let Σ be a Riemann surface with n admissible cylindrical ends.

Let $P \to \Sigma$ be a principal G-bundle.

Definition 2.2.2. A *framing* on the i-th end of P is a principal G-bundle $P_i \to S^1$ and an isomorphism of P with $\pi^* P_i$ along the i-th end, where $\pi : S^1 \times \mathbb{R} \to S^1$ is the projection.

Suppose that P is equipped with a framing on each end. We denote by $\mathcal{A}(P)$ the space of smooth connections on P with asymptotic limits at infinity along each end, that is, there exists a connection $\lim_i(A)$ on P_i such that

$$\lim_i (A)(\theta_i) = \lim_{r_i \to \infty} A(r_i, \theta_i), \quad i = 1, \ldots, n. \tag{3}$$

Let X be a compact, connected Hamiltonian G-manifold with moment map $\Phi : X \to \mathfrak{g}^*$. Consider the associated bundle

$$\pi : P(X) = (P \times X)/G \to \Sigma.$$

Let $\Gamma(\Sigma, P(X))$ denote the space of asymptotically constant sections of $P(X)$, that is, sections u with a constant limit

$$\lim_i (u) := \lim_{r_i \to \infty} u_i(r_i, \theta_i)$$

along each end. I we denote by z_i the point at infinity in each cylindrical end, then let $\overline{u}$ be the extension of u so that $\overline{u}(z_i) = \lim_i(u)$. Let

$$\mathcal{A}(P) \times \mathcal{J}(X)^G \to \mathcal{J}(P(X)), \quad (A, J) \mapsto J_A$$

denote the map which assigns to any connection and invariant almost complex structure on X the almost complex structure on $P(X)$ determined by the splitting induced by A.

We define a suitable notion of energy of a pair (A, u) as follows. Equip Σ with the metric with cylindrical form on the ends. We wish to consider vortices for area forms with suitable exponential decay on the cylindrical ends.

Definition 2.2.3. An area form $\omega_\Sigma \in \Omega^2(\Sigma)$ is *admissible* if and only if on each cylindrical end, with coordinates s, t, there exists constants $c, C > 0$ such that the following estimates hold: if $\omega_\Sigma = \lambda^2(s,t) ds \wedge dt$ then

$$|\lambda(s,t)| \leq C e^{-cs}, \qquad 9|d\lambda|^2 - \Delta(\lambda^2) \leq c$$

check ....

In particular, the form $e^{-cs} ds \wedge dt$ is admissible for $c > 0$.

Let $X, \omega, G, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \Phi$ be as above, Σ be a real surface (not necessarily orientable), P a principal G-bundle over Σ, $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_\Sigma$ be Riemannian metrics on X and Σ, and $(A, u) \in \mathcal{A}(P) \times \Gamma(P(X))$. We define the *energy density* of (A, u) to be the function

$$e_{(A,u)} := e_{(A,u)}^{\langle \cdot,\cdot \rangle_X, \langle \cdot,\cdot \rangle_\Sigma} := \frac{1}{2}\big(|\mathrm{d}_A u|^2 + |F_A|^2 + |u^* P(\Phi)|^2\big) : \Sigma \to \mathbb{R},$$

where the norms are w.r.t. the metrics $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_\Sigma$. Furthermore, we define the *energy* of (A, u) to be the integral

$$E(A, u) := \int_\Sigma e_{(A,u)} d\mu_{\langle \cdot,\cdot \rangle_\Sigma},$$

where $\mu_{\langle \cdot,\cdot \rangle_\Sigma}$ is the measure on Σ induced by $\langle \cdot, \cdot \rangle_\Sigma$. As in the case of ordinary pseudo-holomorphic maps, there is a relation of the energy to a suitable notion of *equivariant symplectic area*. Recall that the Cartan construction gives rise to a map

$$\mathcal{A}(P) \times \Omega_G(X) \to \Omega(P(X))$$

given by

$$(A, \beta) \mapsto \pi_{\mathrm{basic}}((p_2^* \beta) \wedge (p_1^* A))$$

where p_1, p_2 are the projections of $P \times X$ on the factors and π_{basic} is projection on the subspace of basic forms. Applying this to the equivariant symplectic form $\omega_G \in \Omega^2_G(X)$, for each A we obtain a closed two-form

$$\omega_{P(X),A} \in \Omega^2(\Sigma, P(X)), \qquad \pi^* \omega_{P(X),A} = p_2^* \omega_G + \mathrm{d}(p_1^* A, p_2^* \Phi).$$

Definition 2.2.4. The *equivariant symplectic area* of $(A, u) \in \mathcal{A}(P) \times \Gamma(P(X))$ is

$$D(A, u) := \int_\Sigma u^*(\omega_{P(X),A})$$

if finite.

Definition 2.2.5. Given $J_X \in \mathcal{J}(X)^G$, for each section $u \in \Omega^0(\Sigma; P(X))$ define the operator

$$\overline{\partial}_A(u) := \frac{1}{2}(\mathrm{d}u + J_A(u) \circ \mathrm{d}u \circ j) \in \Omega^{0,1}(\Sigma; u^* T^{\mathrm{vert}} P(X)).$$

Lemma 2.2.6. *The energy and equivariant symplectic area of a pair (A, u) are related by*

$$E(A, u) = D(A, u) + \int_\Sigma \left(|\overline{\partial}_A u|^2 + \frac{1}{2}|F_A + u^* P(\Phi) \omega_\Sigma|^2 \right) \omega_\Sigma. \tag{4}$$

Proof. As in [2, Proposition 3.1]. □

Definition 2.2.7. A *gauged holomorphic map* is a pair $(A, u) \in \mathcal{A}(P) \times \Gamma(P(X))$ with $\overline{\partial}_A u = 0$. Let $\mathcal{A}(P, X)$ denote the space of gauged holomorphic maps of finite energy.

Later it will be useful to consider $u : \Sigma \to P(X)$ as a pseudoholomorphic map to $P(X)$. The latter can be given a symplectic form as follows. Let $\omega_\Sigma \in \Omega^2(\Sigma)$ be an area form on Σ. For any $c > 0$ let

$$\omega_{A,c} = \omega_A + c\pi^*\omega_\Sigma \in \Omega^2(P(X)). \tag{5}$$

Lemma 2.2.8. *Let Σ, X be compact. For any $c_1 > 0$, there exists a $c_2 > 0$ such that if $\sup |A|_{C_1} < c_1$ and $c > c_2$ then $\omega_{A,c} \in \Omega^2(P(X))$ is symplectic.*

Proof. It suffices to add on a sufficiently large multiple of $\pi^*\omega_\Sigma$ so that $\omega_{A,c}$ is positive on the horizontal subspace. Since the norm of ω_A on the horizontal subspace depends linearly on the C^1-norm of the connection and the moment map, the claim follows. □

The almost complex structure J_A determined by $J \in \mathcal{J}(X)^G$ and the connection is automatically compatible with $\omega_{A,c}$. We denote by $g_{A,c}$ the metric determined by $J_A, \omega_{A,c}$ on $P(X)$, and $\| \cdot \|_{A,c}$ the corresponding norm. For sections u of $P(X)$ define

$$E_{A,c}(u) = \int_\Sigma g_{A,c}(\mathrm{d}u, \mathrm{d}u), \qquad D_{A,c}(u) = \int_\Sigma u^*\omega_{A,c}$$

denote the corresponding energy and equivariant symplectic area, so that the energy-action relation for pseudoholomorphic u holds

$$E_{A,c}(u) = D_{A,c}(u).$$

Definition 2.2.9. A *gauged holomorphic map with fixed holonomies* $\underline{\mu} \in \mathfrak{A}^n$ is a pair $(A, u) \in \mathcal{H}(P, X)$ with $\lim_i A = \mu_i \mathrm{d}\theta_i$ for each cylindrical end $i = 1, \ldots, n$ as in (3). Denote by $\mathcal{H}(P, X, \underline{\mu})$ the space of gauged holomorphic maps with fixed holonomies $\underline{\mu}$.

2.3. **Vortices with fixed holonomy.** We define a moduli space of symplectic vortices using the symplectic quotient construction for the gauge group action on $\mathcal{H}(P, X, \underline{\mu})$. Define a formal (possibly degenerate) closed two-form on $\mathcal{H}(P, X, \underline{\mu})$ by

$$(\xi_1, \xi_2) \mapsto \int_\Sigma u^*\omega_{P(X)}(\xi_1, \xi_2)\omega_\Sigma. \tag{6}$$

Let $\mathcal{G}(P)$ denote the group of gauge transformations of P with limits at infinity, that is, there exists a gauge transformation $\lim_i g \in \mathcal{G}(P_i)$ with

$$\lim_i g(\theta_i) = \lim_{r_i \to \infty} g(\theta_i, r_i).$$

$\mathcal{G}(P)$ naturally acts on $\mathcal{H}(P, X)$ preserving the two-form (6). For any $\xi \in \mathfrak{g}(P)$ we denote by

$$\xi_X \in \Omega^0(\Sigma, P(\mathrm{Vect}(X)))$$

the $P(\mathrm{Vect}(X))$-valued function generated by the action. The generating vector field for the action of ξ is given by

$$u^*\xi_X \in \Omega^0(\Sigma, T^{\mathrm{vert}}P(X)), \quad (u^*\xi_X)(z) = \xi_X(z)(u(z)).$$

Let ∂P denote the bundle at infinity

$$\partial P = \bigcup_{i=1}^{n} P_i.$$

The action of $\mathcal{G}(P)$ on $\mathcal{H}(P,X)$ is formally Hamiltonian with moment map given by

$$\mathcal{H}(P,X) \to \Omega^2(\Sigma, P(\mathfrak{g})) \oplus \mathcal{A}(\partial P), \quad (A,u) \mapsto (F_A + \omega_\Sigma u^* P(\Phi), A|_{\partial P}).$$

(Note that the moment map equation makes sense for affine target.) These formal considerations motivate the following definition. Let $\underline{\mu} \in \mathfrak{A}^n$ be a n-tuple of elements of the Weyl alcove, parametrizing a gauge orbit in $\mathcal{A}(\partial P)$. Let $\mathcal{G}(P,\underline{\mu})$ denote the subgroup of $\mathcal{G}(P)$ fixing the connections $\mu_i \mathrm{d}\theta_i$ at infinity.

Definition 2.3.1. A *symplectic vortex with fixed holonomy* is a pair $(A,u) \in \mathcal{H}(P,X,\underline{\mu})$ with $F_A + \omega_\Sigma u^* P(\Phi) = 0$. An isomorphism of vortices $(A_j,u_j), j = 0,1$ is a gauge transformation $g \in \mathcal{G}(P,\underline{\mu})$ with $g(A_0,u_1) = (A_1,u_1)$. Denote by $\mathcal{A}_0(P,X,\underline{\mu})$ the set of symplectic vortices with fixed holonomies $\underline{\mu}$. A vortex is *stable* if it has finite automorphism group. Let $M(P,X,\underline{\mu})$ denote the moduli space of isomorphism classes vortices with fixed holonomy,

$$M(P,X,\underline{\mu}) := \mathcal{A}_0(P,X,\underline{\mu}) /\!\!/ \mathcal{G}(P,\underline{\mu}) \tag{7}$$

and $M(\Sigma,X,\underline{\mu})$ the union over topological types of bundles $P \to \Sigma$.

In order to define evaluation maps, it is necessary to introduce framed symplectic vortices.

Definition 2.3.2. A *framing* of P at infinity at the i-th end is a trivialization $\phi_i : \overline{P}_i \to G$. A *framed vortex* is a vortex (A,u) with holonomies at infinite $\underline{\mu} = (\mu_1,\ldots,\mu_n)$ is a vortex (A,u) equipped with framings at infinity for each cylindrical end such that

$$\phi_i : \overline{P}_i^{\mathrm{Hol}_i(A)} \to G_{\mu_i}.$$

An *isomorphism* of framed vortices is an isomorphism of vortices intertwining the framings.

Let $M^{\mathrm{fr}}(\Sigma,X,\underline{\mu})$ denote the space of framed vortices with fixed holonomy at infinity resp. moduli space of isomorphism classes of framed vortices with fixed holonomy. The evaluation maps at infinity

$$[A,u] \mapsto (\overline{u}(z_i))$$

are gauge-invariant and descend to maps

$$\mathrm{ev}^{\mathrm{fr}} : M^{\mathrm{fr}}(\Sigma,X,\underline{\mu}) \to X^n.$$

We will see later that the finite energy condition forces the evaluation map ev to take values in the fixed point set of the limit of the connection. That is, if $\lim_i A = \mu_i \mathrm{d}\theta_i$ then

$$\mathrm{ev}_i^{\mathrm{fr}}(A,u) \in X^{\mu_i}, \quad i = 1,\ldots,n.$$

Consider the map

$$\pi : M^{\mathrm{fr}}(\Sigma, X, \underline{\mu}) \to M(\Sigma, X, \underline{\mu})$$

given by forgetting the framings. Suppose that every vortex is stable with trivial stabilizer, and suppose that there exists a classifying map

$$\psi : M^{\mathrm{fr}}(\Sigma, X, \underline{\mu}) \to EG_{\underline{\mu}} := \prod_{i=1}^{n} EG_{\mu_1} \times \ldots \times EG_{\mu_n}$$

for the bundle π. Combining the classifying map with the evaluation maps above gives a map

$$\mathrm{ev} : M(\Sigma, X, \underline{\mu}) \to X^{n,\underline{\mu}}_{G_{\underline{\mu}}} := \prod_{i=1}^{n} X^{\mu_i} \times_{G_{\mu_i}} EG_{\mu_i}.$$

More generally, the evaluation map exists rationally on the stable locus, assuming the existence of a classifying map. That is, we have a map

$$\mathrm{ev}^*_{\mathbb{Q}} : \bigotimes_{i=1}^{n} H_{G_{\mu_i}}(X^{\mu_i}, \mathbb{Q}) \to H(M(\Sigma, X, \underline{\mu}), \mathbb{Q}).$$

2.4. **Correspondence with parabolic vortices.** In this section we describe the holomorphic objects corresponding to vortices. Suppose that X is a $G_{\mathbb{C}}$-variety, and Σ is a compact Riemann surface. Recall that a morphism from Σ to the quotient stack $X/G_{\mathbb{C}}$ consists of a pair $(E_{\mathbb{C}}, u)$ of a holomorphic principal $G_{\mathbb{C}}$-bundle $E_{\mathbb{C}} \to \Sigma$ together with a holomorphic section $u : \Sigma \to E_{\mathbb{C}}(X)$.

Suppose now that Σ has cylindrical ends. Let $\overline{\Sigma}$ denote the associated closed surface, obtained by adding points at infinity $z_1, \ldots, z_n$.

Definition 2.4.1. A *parabolic holomorphic map* from Σ to $X/G_{\mathbb{C}}$ consists of a parabolic bundle $\overline{E}_{\mathbb{C}} \to \overline{\Sigma}$ (see Section 2.1) with markings at $z_1, \ldots, z_N$ together with a holomorphic section $u : \overline{\Sigma} \to \overline{E}_{\mathbb{C}}(X)$.

Theorem 2.4.2. *There exists a one-to-one correspondence between pseudoholomorphic gauged maps (E, A, u) with holonomies μ_j and limits along the j-th cylindrical end in $X^{\mu_j}_k$ and parabolic holomorphic maps $(\overline{E}_{\mathbb{C}}, \overline{u})$ with $\overline{u}(z_j)$ in $P_{\mu_j} X^{\mu_j}_k$.*

Proof. Given a connection on a principal G-bundle $E \to \Sigma$ with fixed holonomies $\mu_1, \ldots, \mu_n$ around the ends, one obtains a parabolic $G_{\mathbb{C}}$ bundle $E_{\mathbb{C}} \to \Sigma$ by assigning to each fiber $(E_{\mathbb{C}})_{z_i}$ at infinity the parabolic reduction determined by μ_i, see e.g. [11], by gluing in trivial bundles using the twistings (2). By removal of singularities, a holomorphic section $u : \Sigma \to E_{\mathbb{C}}(X)$ of finite energy extends automatically to a section $\overline{u} : \overline{\Sigma} \to \overline{E}_{\mathbb{C}}(X)$. The value of $\overline{u}$ at the points $z_1, \ldots, z_n$ is described as follows. Let $B \subset G_{\mathbb{C}}$ denote the Borel subgroup whose Lie algebra contains the positive root spaces. Let P_{μ_j} be the parabolic determined by μ_j,

$$P_{\mu_j} = \{g \in G_{\mathbb{C}} | \lim_{t \to \infty} \mathrm{Ad}(\exp(t\mu_j))g \in B\}$$

whose Lie algebra is that of B plus the sum of negative root spaces vanishing on μ_j. The map $x \mapsto \exp(t\mu_j)x$ restricts on each P_{μ_j}-orbit on X to a retraction of $P_{\mu_j}x$ to $P_{\mu_j}x \cap X^{\mu_j}$; in particular, each P_{μ_j} orbit contains an element of X^{μ_j}. We write X^{μ_j} in terms of components.

$$X^{\mu_j} = \bigcup_k X_k^{\mu_j}, \qquad X = \bigcup_k P_{\mu_j} X_k^{\mu_j}$$

where $X_k^{\mu_j}$ are the connected components of X^{μ_j}. In the trivializations at the punctures, the section $\overline{u}$ is given by

$$\exp(N \ln(z)\mu_j/2\pi i)u(z), z \neq 0.$$

Thus $u(0)$ lies in $X_k^{\mu_j}$ if and only if $\overline{u}(0)$ takes values in $P_{\mu_j}X_k^{\mu_j}$, that is, the limit of $\overline{u}(0)$ under the flow defined by μ_j is $u(0)$. This shows □

Recall that S-equivalence is the equivalence relation defined by orbit-closure. Let X be a projectively embedded $G_{\mathbb{C}}$-variety. The Fubini-Study form on X makes X into a Hamiltonian G-manifold.

Definition 2.4.3. A holomorphic map $u : \Sigma \to X/G_{\mathbb{C}}$ is *polystable* if it is complex gauge equivalent to vortex, *stable* if it is complex gauge equivalent to a stable vortex, and *semistable* if it is S-equivalent to a stable vortex.

The definition of semistability depends on the choice of volume form ω_Σ. In particular, if we choose ω_Σ identically zero, then the stable parabolic holomorphic maps are those whose underlying parabolic bundles are parabolic semistable, by the Mehta-Seshadri theorem [8]. The following is now essentially tautological:

Proposition 2.4.4. *There exists a one-to-one correspondence between equivalence classes of polystable parabolic holomorphic maps $\Sigma \to X/G_{\mathbb{C}}$ and symplectic vortices from Σ to X.*

It would be interesting to investigate the Hilbert-Mumford criterion for stability more generally in this context. Mundet [?] has described the answer in the case without markings.

3. Bi-invariant symplectic action

In this section we introduce and study the *bi-invariant symplectic action*, which is a generalization of the *invariant symplectic action* introduced by Gaio-Salamon [4], and further studied in Ziltener [?]. The bi-invariant action of a triple (x, a, a_0) measures the equivariant symplectic area needed to "fill in" a gauged loop (x, a) with a cylinder whose asymptotic limit is a covariant constant loop with respect to the connection a_0. The results will be used for *two* types of "bubbles connect" results: namely, for sphere bubbles and bubbles on the cylindrical ends.

Let (X,ω) be a symplectic manifold, G be a compact connected Lie group acting on X in a Hamiltonian way, with equivariant moment map $\Phi : X \to \mathfrak{g}^*$, $\langle\cdot,\cdot\rangle_X$ be a G-invariant Riemannian metric on X, and $\pi : Q \to S^1 \simeq \mathbb{R}/\mathbb{Z}$ be a (necessarily trivial) principal G-bundle. Consider a pair (x, a_0), where $x : S^1 \to Q(X) := (Q \times X)/G$ is a smooth section and $a_0 \in \mathcal{A}(Q)$ is a connection one-form. We denote by

$$\ell(x, a_0) := \int_{S^1} |d_{a_0} x| \, dt$$

its (twisted) length w.r.t.the metric $\langle\cdot,\cdot\rangle_X$.

Definition 3.0.5. An a_0-*compatible extension* of x is a smooth section $u : [0,1] \times S^1 \to [0,1] \times Q(X)$ such that

$$u(1,\cdot) = x, \quad d_{a_0}(u(0,\cdot)) = 0. \tag{8}$$

We denote by $\mathrm{Ext}_{a_0}(x)$ the set of such extensions and call (x, a_0) *admissible* iff there exists $u \in \mathrm{Ext}_{a_0}(x)$ such that the following holds. If $\hat{u} \in \mathrm{Ext}_{a_0}(x)$ is another extension satisfying

$$\max_{s\in[0,1]} \ell\big(\hat{u}(s,\cdot), a_0\big) \le \max_{s\in[0,1]} \ell\big(u(s,\cdot), a_0\big) \tag{9}$$

then

$$\int_{[0,1]\times S^1} \omega\big(\partial_s \hat{u}, d_{a_0}(\hat{u}(s,\cdot))\cdot\big) \wedge ds = \int_{[0,1]\times S^1} \omega\big(\partial_s u, d_{a_0}(u(s,\cdot))\cdot\big) \wedge ds. \tag{10}$$

Definition 3.0.6. The bi-invariant symplectic action is the map

$$\mathcal{A} : \big\{(x, a, a_0) \,\big|\, (x, a_0) \text{ admissible}, \, a \in \mathcal{A}(Q)\big\} \to \mathbb{R}$$

defined by

$$\mathcal{A}(x, a, a_0) := \int_{[0,1]\times S^1} \omega\big(\partial_s u, d_{a_0}(u(s,\cdot))\cdot\big) \wedge ds + \int_{S^1} \big\langle Q(\Phi)\circ x, a - a_0\big\rangle, \tag{11}$$

where u is as in the definition of admissibility, and we view $a - a_0$ as a one-form on S^1 with values in the adjoint bundle $Q(\mathfrak{g})$.

Remark 3.0.7. Sections $x : S^1 \to Q(X)$ and $u : [0,1] \times S^1 \to Q(X)$ of the bundles are in natural bijection with equivariant maps $Q \to X$ and $[0,1] \times Q \to X$ respectively. We will sometimes use this identification.

Remark 3.0.8. The following gives a more intrinsic but less explicit definition of the action. Let $A \in \mathcal{A}([0,1] \times Q)$ be a connection one-form on the bundle $[0,1] \times Q$. The form

$$\tilde{\omega}_A = \pi_3^*\omega + \mathrm{d}\big\langle \pi_3^*\Phi, \pi_2^* A\big\rangle \in \Omega^2([0,1] \times Q \times X)$$

where π_2, π_3 are the projections on Q, X, descends to a form $\omega_A \in \Omega^2([0,1] \times Q(X))$.

Now let $A \in \mathcal{A}([0,1] \times Q)$ be a connection one-form with $A|_{s=0} = a_0$ and $A|_{s=1} = a$, and let A_0 denote the pull-back of a_0 to $[0,1] \times Q$ under projection onto the second factor. Let (x, a_0) be an admissible pair and $u \in \mathrm{Ext}_{a_0}(x)$ be an extension as in the definition of admissibility. We have

$$\begin{aligned}\mathcal{A}(x, a, a_0) &= \int_{[0,1]\times S^1} u^*\big(\omega_{A_0} + \mathrm{d}\big\langle([0,1]\times Q)(\Phi), A - A_0\big\rangle\big) \\ &= \int_{[0,1]\times S^1} u^*\omega_A.\end{aligned}$$

The following lemma proves in particular that the bi-invariant symplectic action is bi-invariant under gauge transformation. We prove a stronger statement which includes bundle morphisms not necessarily covering the identity. We view such a morphism as a section $\psi : S^1 \to Q(Q) = (Q \times Q)/G$. If $x : S^1 \to Q(X)$ is a section we define the pull-back section $\psi^* x : S^1 \to Q(X)$ by $\psi^* x(t) := \big[q, \tilde{x} \circ \tilde{\psi}(q)\big]$, for every $t \in S^1$, where $q \in Q$ is an arbitrary point over t. Here $\tilde{x} : Q \to X$ and $\tilde{\psi} : Q \to Q$ are the equivariant maps corresponding to x and ψ.

Lemma 3.0.9. *Let X, ω, G, Q and $\langle\cdot,\cdot\rangle_X$ be as above and $\Psi : [0,1] \times Q \to Q$ a smooth equivariant map such that for every $s \in [0,1]$ the following holds. The map $f_s : S^1 \simeq \mathbb{R}/\mathbb{Z} \to S^1$ defined by the equation $f_s \circ \pi = \pi \circ \Psi(s, \cdot)$ satisfies $\frac{d}{dt} f_s(t) \geq 0$, for every $t \in S^1$, or $\frac{d}{dt} f_s(t) \leq 0$, for every $t \in S^1$. Then for every smooth section $x : S^1 \to Q(X)$ and every $a_0 \in \mathcal{A}(Q)$, the pair $\big(x \circ \Psi(1, \cdot), \Psi(0, \cdot)^* a_0\big)$ is admissible if and only if (x, a_0) is, and*

$$\mathcal{A}\big(\Psi(1,\cdot)^* x, \Psi(0,\cdot)^* a_0, \Psi(1,\cdot)^* a\big) = \deg(f_0)\mathcal{A}(x, a, a_0).$$

Proof of Lemma 3.0.9. We denote $d := \deg(f)$.

...

It suffices to construct maps

$$\mathrm{Ext}_{a_0}(x) \to \mathrm{Ext}(\phi^*(x, a_0)), \quad \mathrm{Ext}(\phi^*(x, a_0)) \to \mathrm{Ext}(x, a_0)$$

preserving the quantities in (10), (9). We choose smooth functions $\psi, \chi : [0,1] \to [0,1]$ satisfying

$$\psi(0) = 0,\ \psi(s) = 1,\ \forall s \in [1/2, 1], \quad \chi(s) = 0,\ \forall s \in [0, 1/2],\ \chi(1) = 1. \tag{12}$$

Given an extension u' of $\phi^* x$ we define

$$u(s,t) := \begin{cases} (\zeta_{\psi(s)}^* x)(t) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ ((\chi \times 1)^* u')(s,t) & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{13}$$

That is, on the first region u is equal to the section given by pull-back of x by the map $(s, q) \mapsto \zeta_{\psi(s)}(q)$ and on the second region u is the pull-back of u' by the map $(s, q) \mapsto (\chi(s), q)$. The section u is smooth since $u'(0, \cdot) = \phi^* x = \zeta_1^* x$. If A' is an extension of $a' = \phi^* a$ with limit a_0 as in Remark 3.0.8, define $\rho(s, q) = \zeta_{\psi(s)}(q)$ and

$$A(s,t) := \begin{cases} (\rho^* a_0)(t) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ ((\chi \times 1)^* A')(s,t) & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} \tag{14}$$

We have

$$\max_s \ell(u(s,\cdot),a_0) \le \max_s \ell\big(u'(s,\cdot),a_0\big) \tag{15}$$

since $\frac{d}{dt}\pi(\zeta(s,t)) \ge 0$. On the other hand,

$$\int_\Sigma u^*\omega_A = \int_{[0,1/2]\times S^1} (\rho^* x)^*\omega_{\rho^* A} + \int_{[1/2,1]\times S^1} (\chi\times 1)^*\omega_{A'}.$$

To see that the first integral vanishes, note that $(\rho^* x)\omega_{\rho^* A} = \zeta^* x^* \omega_{a_0}$ where $\zeta(s,t) = \pi(\rho(s,\pi^{-1}(t)))$ is the map covered by ρ. Since ω_{A_0} is degree two, $x^*\omega_{a_0} = 0$. To see that the second integral vanishes, consider the map $\eta(\lambda,s,t) = ((1-\lambda)s+\chi(s),t)$. By Stokes' theorem

$$\begin{aligned}
0 &= \int_{[0,1]\times\Sigma} (\eta^* u')^* \mathrm{d}\omega_{\eta^* A} \\
&= \int_{(\{0,1\}\times\Sigma)\cup([0,1]\times\partial\Sigma)} (\eta^* u')^*\omega_{\eta^* A} \\
&= \int_\Sigma (u')^*\omega_{A'} - \int_\Sigma u^*\omega_A \qquad (16)
\end{aligned}$$

where to obtain the third equality we have used that $\eta([0,1]\times\partial\Sigma) \subseteq \partial\Sigma$ and so the pull-back of $(\rho^* u')\omega_{\rho^* A}$ to $[0,1]\times\partial\Sigma$ vanishes. Hence

$$\int_\Sigma u^*\omega_A = \int_{[0,1]\times S^1} (u')^*\omega_{A'}.$$

Conversely, given an extension $u \in \mathrm{Ext}(x,a_0)$ define

$$u'(s,t) := \begin{cases} (\zeta^*_{\psi(1-s)} x)(t) & \text{if } 0 \le s \le \frac{1}{2}, \\ ((\chi\times 1)^* u)(s,t) & \text{if } \frac{1}{2} \le s \le 1 \end{cases} \tag{17}$$

and with $\rho(s,q) = \zeta_{\psi(1-s)}(q)$

$$A'(s,t) := \begin{cases} (\rho^* a_0)(t) & \text{if } 0 \le s \le \frac{1}{2}, \\ ((\chi\times 1)^* A)(s,t) & \text{if } \frac{1}{2} \le s \le 1. \end{cases} \tag{18}$$

Similar arguments to those before show that the maximal length of u' and the integral of $(u')^*\omega_{A'}$ are the same as those for A, u. This proves Lemma 3.0.9. □

Proposition 3.0.10 (Equivariant isoperimetric inequality). *Let $X, \omega, G, \langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and $\Phi, \langle\cdot,\cdot\rangle_X$ be as above, and let $\mathcal{C} \subseteq G$ be a conjugacy class. Assume that X is closed. Then there exist constants $\delta, C > 0$ such that for every principal G-bundle Q over S^1, every connection $a_0 \in \mathcal{A}(Q)$ with holonomy in $\mathcal{C}$ and every section $x : S^1 \to (Q\times X)/G$ satisfying*

$$\ell(x,a_0) \le \delta \tag{19}$$

the following holds. The pair (x,a_0) is admissible and for every connection $a \in \mathcal{A}(Q)$ and every $\epsilon > 0$ and $1 \le p \le \infty$ the following inequality holds:

$$|\mathcal{A}(x,a,a_0)| \le C\|d_a x\|_p^2 + \left(\frac{1}{4\epsilon} + C\right)\|a - a_0\|_p^2 + \epsilon\|x^* Q(\Phi)\|^2_{\frac{p}{p-1}}. \tag{20}$$

Proposition 3.0.11 (Energy action identity)**.** *Let $X, \omega, G, \langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and Φ be as above, J be a G-invariant ω-compatible almost complex structure, and let $\mathcal{C} \subseteq G$ be a conjugacy class. Assume that X is closed. Then there exists a constant $\delta > 0$ such that the following holds. Let $s_- \leq s_+$ be numbers, Q be a principal G-bundle over S^1, a_0 be a connection on Q with holonomy in $\mathcal{C}$, ω_Σ be an area form on $\Sigma := [s_-, s_+] \times S^1$, and (u, A) be a solution of the vortex equations (**??**) on $[s_-, s_+] \times Q$. Assume that*

$$\ell_{g_{\omega,J}}\big(u(s,\cdot), a_0\big) \leq \delta, \ \forall s \in [s_-, s_+]. \tag{21}$$

Then the pairs $\big(u(s_\pm,\cdot), a_0\big)$ are admissible and

$$E(u, A) = -\mathcal{A}\big((u, A)(s_+,\cdot), a_0\big) + \mathcal{A}\big((u, A)(s_-,\cdot), a_0\big). \tag{22}$$

To prove Propositions 3.0.10 and 3.0.11, we use the symplectic action for paths with Lagrangian boundary conditions. More precisely, let (X, ω) be a symplectic manifold, $L_0, L_1 \subseteq X$ be Lagrangian submanifolds and let $\langle\cdot,\cdot\rangle_X$ be a Riemannian metric on X. Let $x : [0, 1] \to X$ be a smooth path such that $x(i) \in L_i$, for $i = 0, 1$. We denote by $\ell(x) := \int_0^1 |\dot{x}|\, dt$ its length w.r.t.$\langle\cdot,\cdot\rangle_X$.

Definition 3.0.12. An (L_0, L_1)-*compatible extension* of x is a smooth map $u : \Sigma := [0, 1] \times [0, 1] \to X$ such that

$$u(0, t) \in L_0 \cap L_1, \ \forall t, \quad u(1,\cdot) = x, \quad u(s, i) \in L_i, \forall s \in [0, 1],\ i = 0, 1. \tag{23}$$

We denote by $\mathrm{Ext}_{L_0,L_1}(x)$ the set of all such extensions, and we call the path x *admissible* iff there exists $u \in \mathrm{Ext}_{L_0,L_1}(x)$ such that the following holds. If $\hat{u} \in \mathrm{Ext}_{L_0,L_1}(x)$ is another extension satisfying

$$\max_s \ell\big(\hat{u}(s,\cdot)\big) \leq \max_s \ell\big(u(s,\cdot)\big), \quad \forall s \in [0, 1], \tag{24}$$

then

$$\int_\Sigma \hat{u}^*\omega = \int_\Sigma u^*\omega. \tag{25}$$

We define the *relative action* to be the map

$$\mathcal{A}_{L_0,L_1} : \big\{x \in C^\infty([0, 1], X) \,\big|\, x \text{ admissible}\big\} \to \mathbb{R},$$

$$\mathcal{A}_{L_0,L_1}(x) := -\int_\Sigma u^*\omega,$$

where $u : \Sigma \to X$ is a smooth map with the above properties. Admissibility and the relative symplectic action are invariant under changes of coordinates. This is the content of the following.

Lemma 3.0.13. *Let X, ω, L_0, L_1 and $\langle\cdot,\cdot\rangle_X$ be as above, $x : [0, 1] \to X$ a path, and $\phi : [0, 1] \to [0, 1]$ be a smooth map fixing 0 and 1 such that $\phi'(t) \geq 0$, for every $t \in [0, 1]$. Then $x \circ \phi$ is admissible if and only if x is, and*

$$\mathcal{A}_{L_0,L_1}(x \circ \phi) = \mathcal{A}_{L_0,L_1}(x)$$

Proof of Lemma 3.0.13. The proof is similar to that of Lemma 3.0.9. It suffices to construct maps

$$\mathrm{Ext}_{L_0,L_1}(x) \to \mathrm{Ext}_{L_0,L_1}(x \circ \phi), \quad \mathrm{Ext}_{L_0,L_1}(x \circ \phi) \to \mathrm{Ext}_{L_0,L_1}(x)$$

preserving the symplectic area and maximal length in (25), (24). Let ψ, χ be as in (12). Given an extension $u' \in \mathrm{Ext}_{L_0,L_1}(x \circ \phi)$, define $u \in \mathrm{Ext}_{L_0,L_1}(x)$ by

$$u(s,t) := \begin{cases} x((1-\psi(s))t + \psi(s)\phi(t)) & \text{if } 0 \le s \le \frac{1}{2}, \\ u(\chi(s), t) & \text{if } \frac{1}{2} \le s \le 1. \end{cases}.$$

One sees easily that u has the same area and maximal length as u', The reverse construction is similar.

□

Remark 3.0.14. Their are alternative ways of defining admissibility and the relative symplectic action, for example, one could define it as follows. If $x, y \in X$ are points, we denote their distance by $d(x,y)$, if $x : [0,1] \to X$ is a path, we denote its length by $\ell(x)$, and if $S \subseteq X$ is a subset, we denote its diameter by

$$\mathrm{diam}(S) := \sup_{x,y \in S} d(x,y).$$

(All these quantities are with respect to $\langle \cdot, \cdot \rangle_X$.) We denote by $\Sigma := [0,1] \times [0,1]$ the unit square. We call a path $x \in W^{1,\infty}([0,1], X)$ *admissible* iff $x(i) \in L_i$, for $i = 0, 1$, and there exists a map $u \in W^{1,\infty}(\Sigma, X)$ such that

$$u(0,t) = u(0,0),\ \forall t, \quad u(1,\cdot) = x, \quad u(s,i) \in L_i, \forall s \in [0,1],\ i = 0,1 \tag{26}$$

and the following holds. If $\hat{u} \in W^{1,\infty}(\Sigma, X)$ is another map satisfying (26) and the condition $\mathrm{diam}(\hat{u}(\Sigma)) \le \mathrm{diam}(u(\Sigma))$ then

$$\int_\Sigma \hat{u}^*\omega = \int_\Sigma u^*\omega.$$

Proposition 3.0.15 (Relative isoperimetric inequality)**.** *Let X, ω, L_0, L_1 and $\langle \cdot, \cdot \rangle_X$ be as above. Assume that L_0 and L_1 are closed and intersect cleanly. Then there exist constants $\delta, C > 0$ such that the following holds. If $x : [0,1] \to X$ is a path satisfying $x(i) \in L_i$, for $i = 0, 1$ and $\ell(x) < \delta$ then x is admissible and*

$$|\mathcal{A}_{L_0,L_1}(x)| \le C \|\dot{x}\|_2^2. \tag{27}$$

Proof of Proposition 3.0.15. This follows from [10, Lemma 3.4.5] .... □

Proposition 3.0.16 (Relative area action identity)**.** *Let X, ω, L_0 and L_1 be as in Proposition 3.0.15, and let J be an ω-compatible almost complex structure on X. Then there exists $\delta > 0$ such that for every pair $s_- \le s_+$ and every smooth map $u : \Sigma := [s_-, s_+] \times [0,1] \to X$ the following holds. If $u(s,i) \in L_i$, for $i = 0, 1$, and $\ell_{g_{\omega,J}}(u(s,\cdot)) < \delta$, for every $s \in [s_-, s_+]$, then $u(s_-, \cdot)$ and $u(s_+, \cdot)$ are admissible, and*

$$\int_\Sigma u^*\omega = -\mathcal{A}_{L_0,L_1}(u(s_+,\cdot)) + \mathcal{A}_{L_0,L_1}(u(s_-,\cdot)). \tag{28}$$

Proof of Proposition 3.0.16. We set $Y_\pm := L_\pm$ and denote by $i_\pm : L_\pm \to X$ the inclusions. Since $L_\pm$ is Lagrangian the hypotheses of Lemma 5.3.4 below are satisfied. So let U and α be as in the conclusion of that lemma. As in the proof of Lemma 5.3.4 there exists an open neighborhood $V \subseteq U$ of $L_- \cap L_+$ and a strong deformation retraction $r : [0,1] \times V \to L_- \cap L_+$ preserving $L_- \cap V$ and $L_+ \cap V$. Since X, L_+, L_- are compact, there exists a constant $\delta > 0$ such that if $x \in X$ then

$$\max(\operatorname{dist}(x, L_+), \operatorname{dist}(x, L_-)) < \delta \implies x \in V. \tag{29}$$

Let s_-, s_+ and u be as in the hypothesis. We define the map

$$h : [0,1] \times [s_-, s_+] \times [0,1] \to X, \quad h(\lambda, s, t) := r(\lambda, u(s,t)).$$

We fix $s_0 \in [s_-, s_+]$. The conditions (8) are satisfied with x and u replaced by $u(s_0, \cdot)$ and $h(\cdot, s_0, \cdot)$. Hence $h(\cdot, s_0, \cdot) \in \operatorname{Ext}_{L_0,L_1}(u(s_0,\cdot))$. Let $\hat{u} \in \operatorname{Ext}_{L_0,L_1}(u(s_0,\cdot))$ be another extension. By the assertion of Lemma 5.3.4 we have $d\alpha = \omega$ on V and $i_\pm|^*_{V \cap L_\pm}\alpha = 0$. Hence by Stokes' theorem,

$$\begin{aligned}
\int_{[0,1]\times[0,1]} \hat{u}^*\omega &= \int_{\partial\left([0,1]\times[0,1]\right)} \hat{u}^*\alpha \\
&= \int_{[0,1]} \hat{u}(1,\cdot)^*\alpha \\
&= \int_{\partial\left([0,1]\times[0,1]\right)} h(\cdot, s_0, \cdot)^*\alpha \\
&= \int_{[0,1]\times[0,1]} h(\cdot, s_0, \cdot)^*\omega.
\end{aligned}$$

Here in the second equality we used $i_\pm|^*_{V\cap L_\pm}\alpha = 0$ and the first and the third conditions in (8) for $\hat{u}$, in the third step we used the second condition in (8) for $\hat{u}$ and $h(\cdot, s_0, \cdot)$. It follows that $u(s_0, \cdot)$ is admissible. To see that (28) holds, we use again Stokes' theorem, to get

$$\begin{aligned}
0 &= \int_{[0,1]\times[s_-,s_+]\times[0,1]} h^* d\omega \\
&= \int_{[s_-,s_+]\times[0,1]} h(1,\cdot,\cdot)^*\omega - \int_{[0,1]\times[0,1]} h(\cdot, s_+, \cdot)^*\omega + \int_{[0,1]\times[0,1]} h(\cdot, s_-, \cdot)^*\omega \\
&= \int_\Sigma u^*\omega + \mathcal{A}_{L_0,L_1}(u(s_+,\cdot)) - \mathcal{A}_{L_0,L_1}(u(s_-,\cdot)).
\end{aligned}$$

Here in the second equality we used the Lagrangian boundary conditions $u(s,i) \in L_i$, for $i = 0,1$ and $s \in [s_-, s_+]$, and the fact that r preserves L_i. This proves Proposition 3.0.16. □

Lemma 3.0.17. *Let X be a manifold and $\phi : X \to X$ be a diffeomorphism such that X admits a ϕ-invariant metric. Let $Y := \{(x,x) \mid x \in X\} \subset X^2$ denote the diagonal and $Z := \{(x, \phi(x)) \mid x \in X\} \subset X^2$ the graph of ϕ. Then the intersection $Y \cap Z$ is clean.*

Proof. The tangent spaces to Y, Z are

$$TY = \{((x,v),(x,v)), \quad x \in X, v \in T_xX\}$$
$$TZ = \{((x,v),(\phi(x), D_x\phi(v))), \quad x \in X, v \in T_xX\}.$$

Hence

$$Y \cap Z = \{(x,x) \in X^2,\ \phi(x) = x\}$$
$$TY \cap TZ = \{((x,v),(x,v)) \in TX^2,\ (\phi(x), D_x\phi(v)) = (x,v)\}.$$

Projection onto the first factor gives identifications

$$Y \cap Z \to X^\phi, \qquad TY \cap TZ \to (TX)^{D\phi}$$

where X^ϕ resp. $(TX)^\phi$ denotes the fixed point set of the action of ϕ on X resp. TX. Thus the intersection is clean if and only if X^ϕ is a submanifold and

$$T(X^\phi) = (TX)^{D\phi}. \tag{30}$$

Let $x \in Y \cap Z$. Let g be a ϕ-invariant metric on X and denote by $\exp_x : T_xX \to X$ the exponential map. It follows from the uniqueness of geodesics with given initial conditions that

$$\exp_x \circ D_x\phi = \phi \circ \exp_x. \tag{31}$$

We choose a number $r > 0$ such that $\exp_x : B_r \to X$ is a diffeomorphism onto its image, where $B_r \subset T_xX$ denotes the ball of radius r around 0 w.r.t. the metric on T_xX. By equation (31) the fixed point set $B_r \cap T_xX^{D_x\phi}$ is mapped bijectively onto the fixed point set $\exp(B_r) \cap X^\phi$. Because $T_xX^{D_x\phi}$ is a subspace, X^ϕ is a manifold near x. Taking tangent spaces we obtain

$$T_x(X^\phi) = D\exp_x(0)T_0(T_xX)^{D_x\phi} = (T_xX)^{D_x\phi}.$$

Hence (30) which proves the Lemma. □

We now relate the bi-invariant and relative symplectic actions and the Yang-Mills-Higgs and Dirichlet energy. Let $\mathcal{C} \subseteq G$ be a conjugacy class. We choose a representative $g \in \mathcal{C}$. Consider the manifold $\tilde{X} := X \times X$ with the symplectic structure $\tilde{\omega} := (-\omega) \oplus \omega$ and the Riemannian metric $\langle \cdot, \cdot \rangle_{\tilde{X}}$ induced by $\langle \cdot, \cdot \rangle_X$, and the Lagrangian submanifolds

$$L_0 := \big\{(x,x) \,|\, x \in X\big\}, \quad L_1 := \big\{(x, g^{-1}x) \,\big|\, x \in X,\, g \in G\big\}.$$

Let $a_0 \in \mathcal{A}(Q)$ be a connection with holonomy in $\mathcal{C}$ and $x : S^1 \to Q(X)$ be a section satisfying $\ell(x, a_0) \le \delta$. We define $\rho : [0,1] \to S^1 \simeq \mathbb{R}/\mathbb{Z}$ by $\rho(t) := t + \mathbb{Z}$. Since a_0 has holonomy conjugate to g, there exists a map $f : [0,1] \to Q$ satisfying the conditions

$$\pi \circ f = \rho, \quad a_0(\partial_t f) = 0, \quad f(1) = f(0)g. \tag{32}$$

Viewing x as an equivariant map from Q to X, we define

$$\tilde{x} : [0,1] \to \tilde{X}, \quad \tilde{x}(t) := \big(x \circ f((1-t)/2), x \circ f((1+t)/2)\big). \tag{33}$$

Then, denoting by $\tilde{\ell}(\tilde{x})$ the length of $\tilde{x}$ w.r.t.$\langle \cdot, \cdot \rangle_{\tilde{X}}$, we have

$$\tilde{x}(i) \in L_i, \text{ for } i = 0,1, \quad \tilde{\ell}(\tilde{x}) = \ell(x, a_0).$$

Furthermore, by Lemma 3.0.17 with $\phi(x) := gx$, the Lagrangians L_0 and L_1 intersect cleanly.

Lemma 3.0.18. *Let $L_0, L_1, x, \tilde{x}, a_0$ be as above. The path $\tilde{x}$ is admissible if and only if (x, a_0) is, and*

$$\mathcal{A}(x, a, a_0) = \mathcal{A}_{L_0,L_1}(\tilde{x}) + \int_{S^1} \langle x^* Q(\Phi), a - a_0 \rangle.$$

Proof. Let $\phi : [0,1] \to [0,1]$ be a smooth map with

$$\frac{d}{dt}\phi(t) \geq 0, \forall t \in [0,1], \quad \phi(t) = 0, t \leq 1/4, \quad \phi(t) = 1, t \geq 3/4. \tag{34}$$

By Lemma 3.0.13, x is admissible if and only if $x \circ \phi$ is, and the two paths have the same action. Similarly, let $\psi : S^1 \to S^1$ denote the corresponding smooth map, obtained from the identification $S^1 = [0,1]/(0 \sim 1)$. Let $\rho : Q \to Q$ be a lift of ψ preserving a_0; given a single value $\rho(0)$, the remaining values $\rho(t)$ are defined by parallel transport of $\rho(0) = \rho(\psi(0))$ along a path from 0 to $\psi(t)$. By Lemma 3.0.9, $\rho^* x$ is admissible if and only if x is, and $\mathcal{A}(\rho^* x, a, a_0) = \mathcal{A}(x, a, a_0)$.

This reduces to the case that $\tilde{x}$ is constant on $[0, 1/4] \cup [3/4, 1]$, and x is covariant constant on $[-1/4, 1/4] \in S^1$. We claim that $\tilde{x}$ is admissible if and only if x is. To prove this, it suffices to define maps

$$\mathrm{Ext}_{a_0}(x) \to \mathrm{Ext}_{L_0,L_1}(\tilde{x}), \quad \mathrm{Ext}_{L_0,L_1}(\tilde{x}) \to \mathrm{Ext}_{a_0}(x)$$

preserving the maximal length and integrals in the definitions of admissibility, where $\mathrm{Ext}_{a_0}(x)$ resp. $\mathrm{Ext}_{L_0,L_1}(\tilde{x})$ is the set of extensio of x resp. $\tilde{x}$, compatible with a_0 and (L_0, L_1) respectively. Given any extension $u \in \mathrm{Ext}_{a_0}(x)$ define $\tilde{u} \in \mathrm{Ext}_{L_0,L_1}(\tilde{x})$ by pullback under f, with the same maximal length and integral. Conversely, given any extension $\tilde{u} \in \mathrm{Ext}_{L_0,L_1}(\tilde{x})$ define an extension $\tilde{u}_1$ of $\tilde{x}$ by pull-back under the map $1 \times \phi$, where ϕ is the map of (34). Since $\tilde{u}_1$ is constant near $0, 1$, it descends to a smooth map

$$u_1 : [0,1] \times S^1 \to Q, \quad u_1(1, \cdot) = x, \quad \ell(u_1(s, \cdot), a_0) = \ell(\tilde{u}(s, \cdot)).$$

One sees easily that u_1 has the same maximal length and

$$\mathcal{A}(x, a, a_0) = \int_{[0,1] \times S^1} u_1^* \omega_{A_0} = \int_{[0,1] \times [0,1]} \tilde{u}^* \omega.$$

Hence x is admissible if and only if $\tilde{x}$ is and

$$\mathcal{A}(x, a, a_0) = \mathcal{A}(x, a, a_0) + \int_{S^1} \langle x^* Q(\Phi), a - a_0 \rangle = \mathcal{A}_{L_0,L_1}(\tilde{x}) + \int_{S^1} \langle x^* Q(\Phi), a - a_0 \rangle$$

which proves Lemma 3.0.18. $\square$

Proof of Proposition 3.0.10. Let $\tilde{X}, \tilde{\omega}, L_0, L_1, \langle \cdot, \cdot \rangle_{\tilde{X}}$ and $\tilde{\ell}$ be as above. We choose constants $\delta > 0$ and $C_1 := C$ as in Lemma **??**, with $X, \omega, \langle \cdot, \cdot \rangle_X$ and ℓ replaced by $\tilde{X}, \tilde{\omega}, \langle \cdot, \cdot \rangle_{\tilde{X}}$ and $\tilde{\ell}$. We define

$$C_2 := \max \left\{ |\xi_X(x)| \,\middle|\, x \in X,\, \xi \in \mathfrak{g} : |\xi| = 1 \right\}, \quad C := \max \left\{ 2C_1, 2C_2^2 \right\}.$$

Let x, a, a_0 be as in the Proposition, $f : [0,1] \to Q$ be a map satisfying (32) and $\tilde{x} : [0,1] \to X$ be the path defined by (33). By the statement of Lemma **??** the path $\tilde{x}$ is admissible. Hence by Lemma 3.0.18 the pair (x, a_0) is admissible. Furthermore, for every $\epsilon > 0$,

$$
\begin{aligned}
\big|\mathcal{A}(x,a,a_0)\big| &\leq \big|\mathcal{A}_{L_0,L_1}(\tilde{x})\big| + \int_{S^1} \big|\big\langle Q(\Phi)\circ x, a-a_0\big\rangle\big| \\
&\leq C_1 \ell(\tilde{x})^2 + \|Q(\Phi)\circ x\|_2 \|a-a_0\|_2 \\
&\leq C_1 \|d_{a_0}x\|_2^2 + \epsilon\|Q(\Phi)\circ x\|_2^2 + \frac{1}{4\epsilon}\|a-a_0\|_2^2. \qquad (35)
\end{aligned}
$$

Here in the first inequality we used Lemma 3.0.18, in the second inequality we used **??** and Hölder's estimate, and in the last inequality we used that $\ell(\tilde{x}) = \|d_{a_0}x\|_1 \leq \|d_{a_0}x\|_2$. Furthermore,

$$
|d_{a_0}x|^2 = \big|d_a x + (a-a_0)_X(x)\big|^2 \leq 2|d_a x|^2 + 2|(a-a_0)_X(x)|^2. \qquad (36)
$$

Since $|(a-a_0)_X(x)| \leq C_2|a-a_0|$, combining (35) with (36) implies

$$
\big|\mathcal{A}(x,a,a_0)\big| \leq C\|d_{a_0}x\|_2^2 + \left(C + \frac{1}{4\epsilon}\right)\|a-a_0\|_2^2 + \epsilon\|Q(\Phi)\circ x\|_2^2.
$$

This proves Proposition **??**. □

Proof of Proposition 3.0.11. Let $X, \omega, G, \langle\cdot,\cdot\rangle_{\mathfrak{g}}, \Phi, J, \mathcal{C}$ be as in the hypothesis of the proposition. We define $\langle\cdot,\cdot\rangle_X := g_{\omega,J}$ and $\tilde{X}, \tilde{\omega}, L_0, L_1, \langle\cdot,\cdot\rangle_{\tilde{X}}, \rho, f$ and $\tilde{\ell}$ be as on page 17. Then L_0 and L_1 are closed and intersect cleanly. Hence the hypotheses of Proposition 3.0.16 are satisfied, with X, ω and J replaced by $\tilde{X}, \tilde{\omega}$ and $\tilde{J} := (-J) \oplus J : T\tilde{X} \to T\tilde{X}$. We choose a number $\delta > 0$ satisfying the conclusion of this proposition. Let $s_- \leq s_+$ and u be as in the proposition. We denote by u' the equivariant map from $[0,1]\times Q$ to X corresponding to u, and we define

$$
\tilde{u} : \Sigma := [2s_-, 2s_+] \times [0,1] \to \tilde{X}, \quad \tilde{u}(s,t) := \big(u'(s/2, f((1-t)/2), u'(s/2, f((1+t)/2)\big).
$$

It follows that $\tilde{u}(s,i) \in L_i$, for $i = 0,1$, $s \in [s_-, s_+]$. Using the second equality in (32), we get

$$
\tilde{\ell}(\tilde{u}(s,\cdot)) = \ell\big(u(s,\cdot), a_0\big) < \delta, \quad \forall s \in [s_-, s_+].
$$

Hence by the assertion of Proposition 3.0.16, the paths $\tilde{u}(s_-,\cdot)$ and $\tilde{u}(s_+,\cdot)$ are admissible, and

$$
\int_\Sigma \tilde{u}^*\tilde{\omega} = -\mathcal{A}_{L_0,L_1}(\tilde{u}(s_+,\cdot)) + \mathcal{A}_{L_0,L_1}(\tilde{u}(s_-,\cdot)). \qquad (37)
$$

On the other hand,

$$
\begin{aligned}
\tilde{\omega}(\partial_s\tilde{u}, \partial_t\tilde{u}) &= \frac{1}{4}\Big(\omega\Big(\partial_s u'\big(s/2, f((1-t)/2)\big), d(u'(s/2,\cdot))\dot{f}((1-t)/2)\Big) \\
&\quad +\omega\Big(\partial_s u'\big(s/2, f((1+t)/2)\big), d(u'(s/2,\cdot))\dot{f}((1+t)/2)\Big)\Big). \qquad (38)
\end{aligned}
$$

The second equality in (32) implies that $d(u'(s/2,\cdot))df = d_{a_0}(u'(s/2,\cdot))d\rho$. Combining this with (38), we obtain

$$\int_\Sigma \tilde{u}^*\tilde{\omega} = \int_\Sigma \tilde{\omega}(\partial_s\tilde{u}, \partial_t\tilde{u})ds \wedge dt = \int_{[s_-,s_+]\times S^1} \omega\big(\partial_s u, d_{a_0}(u(s,\cdot))\,\cdot\,\big) \wedge ds. \tag{39}$$

We denote by $a_\pm$ the pullback of A by the inclusion $\{s_\pm\}\times Q \to [s_-,s_+]\times Q$. By Lemma 3.0.18 with x and $\tilde{x}$ replaced by $u(s_\pm,\cdot)$ and $\tilde{u}(s_\pm,\cdot)$, the pair $(u(s_\pm,\cdot),a_0)$ is admissible, and

$$\mathcal{A}\big(u(s_\pm,\cdot),a_\pm,a_0\big) = \mathcal{A}_{L_0,L_1}(\tilde{u}(s_\pm,\cdot)) + \int_{S^1} \big\langle u(s_\pm,\cdot)^*Q(\Phi), a_\pm - a_0\big\rangle.$$

Substituting this into (37) and combining with (39), we obtain

$$\begin{aligned}
&-\mathcal{A}\big(u(s_+,\cdot),a_+,a_0\big) + \mathcal{A}\big(u(s_-,\cdot),a_-,a_0\big)\\
&= \int_{[s_-,s_+]\times S^1} \omega\big(\partial_s u, d_{a_0}(u(s,\cdot))\,\cdot\,\big)\wedge ds\\
&\quad - \int_{S^1}\big\langle u(s_+,\cdot)^*Q(\Phi), a_+ - a_0\big\rangle + \int_{S^1}\big\langle u(s_-,\cdot)^*Q(\Phi), a_- - a_0\big\rangle.
\end{aligned} \tag{40}$$

We denote by A_0 the pull-back of a_0 to $[0,1]\times Q$ under the projection onto the second factor, and we define $\omega_A \in \Omega^2([0,1]\times Q(X))$ as in Remark 3.0.8. Using Stokes' theorem, the right hand side of (40) equals

$$\int_{[s_-,s_+]\times S^1} u^*\big(\omega_{A_0} + d\big\langle([s_-,s_+]\times Q)(\Phi), A - A_0\big\rangle = \int_{[s_-,s_+]\times S^1} u^*\omega_A = E(A,u),$$

where in the last equality we used Lemma 2.2.6. This proves Proposition 3.0.11. □

4. Energy concentration

In this section we prove various versions of exponential decay used for compactness in the next section. Namely, for symplectic vortices with sufficiently small energy and connection sufficiently close to a base connection, we show that the energy is concentrated on the ends. (This type of result is also known as an *annulus lemma*, see for example [7].) We prove the results for unperturbed symplectic vortices only; the perturbed case can be reduced to this case by the graph construction in [4, Appendix A].

Let (X,ω) be a symplectic manifold, G be a compact connected Lie group acting on X in a Hamiltonian way, with moment map $\Phi : X \to \mathfrak{g}^*$, $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ be an invariant inner product on $\mathfrak{g}$, and J be a ω-compatible G-invariant almost complex structure on X. We denote by ι^G the injectivity radius of G and by d^G the distance function on G, both w.r.t. the metric on G induced by $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. Furthermore, if $\mathcal{C},\mathcal{C}' \subseteq G$ are conjugacy classes then we define

$$\bar{d}^G(\mathcal{C},\mathcal{C}') := \min\big\{d(g,g')\,\big|\, g\in\mathcal{C},\ g'\in\mathcal{C}'\big\}$$

Then $\bar{d}$ is a distance function that induces the quotient topology on the set of conjugacy classes, see for example Lemma A.10 in [**?**].

Theorem 4.0.19. *Assume that X is closed. Then for every conjugacy class $\mathcal{C} \subseteq G$ there exists a constant $\epsilon_0 > 0$ such that for every constant $K \in \mathbb{R}$ there exists a constant C with the following property. Let $s_0, s_1 \in \mathbb{R}$ be such that $s_1 \geq s_0 + 2$, $P \to \Sigma := [s_0, s_1] \times S^1$ be a principal G-bundle, $\lambda \in C^\infty\big([s_0, s_1] \times S^1, (0,\infty)\big)$, $p > 2$ and (u, A) be a $W^{1,p}$-solution of the vortex equations on P with area form $\omega_\Sigma := \lambda^2 ds \wedge dt$. Assume that*

$$\lambda(s,t) \leq Ke^{-\epsilon_0 \min\{s-s_0, s_1-s\}}, \forall s \in [s_0, s_1],\, t \in S^1, \tag{41}$$

$$\Delta(\lambda^2) \geq 0, \quad \max_{x\in X} |\Phi(x)|^2 \left(\sup_{[s_0,s_1]\times S^1} \frac{9|d\lambda|^2}{\Delta(\lambda^2)} - 1\right) \leq K^2, \tag{42}$$

$$E(u, A) \leq \epsilon_0, \tag{43}$$

$$\bar{d}^G(\mathcal{C}, \mathcal{C}_s) \leq Ke^{-4\epsilon_0 \min\{s-s_0, s_1-s\}}, \forall s \in [s_0, s_1], \tag{44}$$

where $\mathcal{C}_s \subseteq G$ denotes the conjugacy class of the holonomy of A around the circle $\{s\}\times S^1$. Then

$$|d_A u|_0(s,t) \leq Ce^{-\epsilon_0 \min\{s-s_0, s_1-s\}}, \quad \forall (s,t) \in [s_0+1, s_1-1] \times S^1, \tag{45}$$

where the norm on the left hand side is induced by the standard metric on Σ and the metric $g_{\omega,J}$ on X.

Let $X, \omega, G, \langle\cdot,\cdot\rangle_{\mathfrak{g}}, \Phi$ and J be as above, Σ be a real surface equipped with two Riemannian metrics $\langle\cdot,\cdot\rangle_\Sigma$ and $\langle\cdot,\cdot\rangle_\Sigma^0$ and a pair $(u, A) \in \Gamma(P(X)) \times \mathcal{A}(P)$ we define the *energy density of (u, A) w.r.t. $\big(\langle\cdot,\cdot\rangle_\Sigma, \langle\cdot,\cdot\rangle_\Sigma^0\big)$* to be

$$e_{(u,A)}^{\langle\cdot,\cdot\rangle_\Sigma, \langle\cdot,\cdot\rangle_\Sigma^0} := \frac{1}{2}\Big(|d_A u|_0^2 + \lambda^2|\mu\circ u|^2 + \lambda^{-2}|F_A|_0^2\Big),$$

where $\lambda : \Sigma \to (0,\infty)$ is the function defined by the equation $\langle\cdot,\cdot\rangle_\Sigma = \lambda^2\langle\cdot,\cdot\rangle_\Sigma^0$, and the point-wise norms $|\cdot|_0$ and $|\cdot|$ are taken w.r.t. $\langle\cdot,\cdot\rangle_\Sigma^0$ and $\langle\cdot,\cdot\rangle_\Sigma$ respectively. In the case $\Sigma \subseteq \mathbb{C}$ or $\Sigma \subseteq \mathbb{R}\times S^1$ and $\langle\cdot,\cdot\rangle_\Sigma^0 = ds^2 + dt^2$, we abbreviate

$$e_{(u,A)}^\lambda := e_{(u,A)}^{\langle\cdot,\cdot\rangle_\Sigma, ds^2+dt^2}.$$

Lemma 4.0.20 (Point-wise bound on energy density)**.** *Let $X, \omega, G, \langle\cdot,\cdot\rangle_{\mathfrak{g}}, \Phi$ and J be as above. Assume that X is closed. Then there exists a constant $\epsilon > 0$ with the following property. Let $K \in \mathbb{R}$, $r > 0$, $\lambda \in C^\infty\big(B_r, (0,\infty)\big)$, $P \to B_r$ be a principal G-bundle, $p > 2$ and (u, A) be a $W^{1,p}$-solution the vortex equations on P with area form $\lambda^2 ds\wedge dt$. Assume that the inequalities (42) and*

$$E^\lambda(A, u) + K^2 \int_{B_r} \lambda^2 \mathrm{d}s\, \mathrm{d}t \leq K\epsilon \tag{46}$$

are satisfied. Then

$$\big(e_{(A,u)}^\lambda + K^2\lambda^2\big)(0) \leq \frac{8}{\pi r^2}\left(E^\lambda(A, u) + K^2 \int_{B_r} \lambda^2 \mathrm{d}s\, \mathrm{d}t\right). \tag{47}$$

Remark 4.0.21. The additional terms $K^2\lambda^2$ and $\frac{8}{\pi r^2}K^2 \int_{B_r} \lambda^2 \mathrm{d}s\, \mathrm{d}t$ in inequality (47) are needed. They may be viewed as the horizontal contribution to the energy density resp.

energy of u viewed as a holomorphic map to $P(X)$ equipped with the metric obtained from the sum of ω_A and the pull-back of $K^2\lambda^2 \mathrm{d}s\mathrm{d}t$.

For the proof of Lemma 4.0.20 we need the following result, which is Lemma 4.3.2. [**?**].

Lemma 4.0.22 (Mean value inequality)**.** *Let $r > 0$ and $C \geq 0$. If $f : B_r \to \mathbb{R}$ is a C^2-function that satisfies the inequalities*

$$f \geq 0, \qquad \Delta f \geq -Cf^2, \qquad \int_{B_r} f < \frac{\pi}{8C}$$

then

$$f(0) \leq \frac{8}{\pi r^2} \int_{B_r} f.$$

Proof of Lemma 4.0.20. Let $X, \omega, G, \langle\cdot,\cdot\rangle_{\mathfrak{g}}, \Phi$ and J be as above, and assume that X is compact. By the calculation in the proof of the pointwise bound on the energy density [**?**, Lemma ?] there exists a constant $C_0 \geq 0$ such that for every $r > 0$, $\lambda \in C^\infty\big(B_r, (0,\infty)\big)$, every principal G-bundle P over B_r and every λ-vortex (A,u) on P we have

$$\Delta e^\lambda_{(A,u)} \geq \Big(-9|d\lambda|^2 + \Delta(\lambda^2) - C_0\lambda^4 \Big)|\mu \circ u|^2 - C_0\big(e^\lambda_{(A,u)}\big)^2. \tag{48}$$

Let now $K \in \mathbb{R}$.

with K replaced by

$$K' := \max\big\{K + C_0K^2, C_0\big\}.$$

Let $r > 0$, $\lambda \in C^\infty\big(B_r, (0,\infty)\big)$, $P \to B_r$ be a principal G-bundle, $p > 2$ and (u, A) be a $W^{1,p}$-solution the vortex equations on P with area form $\lambda^2 ds \wedge dt$. Assume that the conditions (**??**,**??**,46) are satisfied. By elliptic regularity and invariance of energy density under gauge transformation **??** the density $e^\lambda_{(A,u)}$ is smooth. Furthermore, the first inequality in (**??**) holds by the definition of $e^\lambda_{(A,u)}$, and the second inequality with K replaced by K' follows from (48,**??**,**??**) and the fact $\lambda^2|\mu\circ u|^2 \leq e^\lambda_{(A,u)}$. Finally, the third inequality from (46). Therefore, inequality (47) follows from the statement of 4.0.22. This proves Lemma 4.0.20. □

Lemma 4.0.23. *Let G be a compact Lie group and $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ be an invariant inner product on $\mathfrak{g} := \mathrm{Lie}(G)$, Q be a principal G-bundle over $S^1 \simeq \mathbb{R}/\mathbb{Z}$, X be a manifold (possibly with boundary), $a_0 \in \mathcal{A}(Q)$ and a be a smooth section of the real vector bundle*

$$\big\{(x,q,\phi)\,\big|\,(x,q) \in X \times Q,\ \phi : T_qQ \to \mathfrak{g} \text{ equivariant, } \phi(q\xi) = \xi,\ \forall \xi \in \mathfrak{g}\big\} \to X \times Q.$$

Denoting by $\mathcal{C}$ and $\mathcal{C}_x$ the conjugacy classes of the holonomy of a_0 and $a(x,\cdot)$ around S^1, we assume that

$$\bar{d}^G(\mathcal{C}, \mathcal{C}_x) < \iota^G.$$

Then there exists an equivariant diffeomorphism $g : X \times Q \to Q$ that induces the canonical projection $X \times S^1 \to S^1$ and satisfies

$$\big|\big(a(x,\cdot,\cdot)\mathrm{d}(g(x,\cdot)) - a_0\big)(q)\big| = \bar{d}^G(\mathcal{C}, \mathcal{C}_x), \tag{49}$$

for every $x \in X$, $q \in Q$.

Remark 4.0.24. Philosophically, in this lemma a is a family of connections on Q and g is a family of gauge transformations on Q, both depending smoothly on $x \in X$. In the case in which X is a point the lemma says that given two connections on $Q \to S^1$ we may gauge transform one of them in such a way that it lies at distance from the other connection given by the distance of the conjugacy classes. For a general manifold X, the lemma can be seen as a parametrized version of this.

Proof of Lemma 4.0.23.

Claim 4.0.25. There exists smooth map $\widetilde{g} : X \times [0,1] \to G$ such that

(50)

Proof of Claim 4.0.25. ...

This proves Claim 4.0.25. □

...

This proves Lemma 4.0.23. □

Proof of Theorem 4.0.19. For $\xi \in \mathfrak{g}$ and $x \in X$ we denote by $\xi_X(x) \in T_xX$ the infinitesimal action of ξ at x, and we define

$$C_1 := \max\big\{|\xi_X(x)| \,\big|\, x \in X,\, \xi \in \mathfrak{g} :\ |\xi| = 1\big\}.$$

Let $\mathcal{C} \subseteq G$ be a conjugacy class. We fix constants $\delta > 0$ and $C_2 := C$ as in Proposition 3.0.10. Shrinking δ if necessary, we may assume that it satisfies the condition of Proposition 3.0.11. We define

$$\epsilon_0 := \min\left\{\frac{1}{4C_2}, \dots\right\}. \tag{51}$$

Let $K \geq 0$. We choose constants $C_3 := C$ and $\tilde{E} := E > 0$ as in Lemma 4.0.20, with K replaced by $\tilde{K} := \min\{K, \sqrt{\epsilon}\}$. We define

$$E := \min\left\{\frac{\tilde{E}}{4}, \frac{\delta^2}{16C_3}\right\}. \tag{52}$$

Let $0 < c < c_0$, $s_0, s_1 \in \mathbb{R}$ be such that $s_1 \geq s_0 + 2$, $P \to \Sigma := [s_0, s_1] \times S^1$ be a principal G-bundle, $\lambda \in C^\infty\big([s_0, s_1] \times S^1, (0,\infty)\big)$, $p > 2$ and (u, A) be a $W^{1,p}$-solution of the vortex equations on P with area form $\omega_\Sigma := \lambda^2 ds \wedge dt$. Assume that conditions (**??**,43,**??**) are satisfied. Let $(s,t) \in [s_0+1, s_1-1] \times S^1$. Consider the map $\phi : B_{\frac{1}{2}} \to [s_0, s_1] \times S^1$ given by $\phi(s', t') := (s + s', t + t')$. We define $\tilde{\lambda} := \lambda \circ \phi : B_{1/2} \to \mathbb{R}$. By the assumptions (**??**) the conditions (**??**) and (**??**) hold with $r := 1/2$ and λ, K replaced by $\tilde{\lambda}, \tilde{K}$. Moreover, by (43) and (52) the condition (46) is satisfied with E, P, A, u replaced by $\tilde{E}$ and $(\tilde{P}, \tilde{A}, \tilde{u}) := \phi^*(P, A, u)$. It follows that the inequality (47) holds with C replaced by C_3, and hence

$$\begin{aligned} |d_A u|(s,t)^2 &\leq e^{\lambda}_{(u,A)}(s,t) \\ &= e^{\tilde{\lambda}}_{(\tilde{A},\tilde{u})}(0,0) \\ &\leq 4C_3 E^{\tilde{\lambda}}(\tilde{A}, \tilde{u}) \\ &\leq 4C_3 E^{\lambda}(A, u). \end{aligned} \tag{53}$$

We define

$$C_4 :=$$

and the function $\big[1, (s_1 - s_0)/2 - 1\big] \to [0, \infty)$ by

$$f(s) := E\big(u, A, \big[s_0 + s, s_1 - s\big] \times S^1\big) + C_4 e^{-4\epsilon_0 s}. \tag{54}$$

Claim 4.0.26. We have $f'(s) \leq -2\epsilon_0 f(s)$, for every $s \in \big[1, (s_1 - s_0)/2 - 1\big]$.

Proof of Claim 4.0.26. We may assume without loss of generality that there exists a principal G-bundle Q over S^1 such that $P = [s_0, s_1] \times Q$. We fix a connection a_0 on Q whose holonomy lies in $\mathcal{C}$.

Claim 4.0.27. There exists a gauge transformation $g : P \to P$ such that

$$\ell\big(gu(s, \cdot), a_0\big) \leq \delta, \tag{55}$$

for every $s \in [s_0, s_1]$.

Proof of Claim 4.0.27. Denoting by $\mathcal{C}_s$ the conjugacy class of the holonomy of $A|_{\{s\} \times Q}$ inequality (44) ... implies that $\bar{d}^G(\mathcal{C}, \mathcal{C}_s) \leq \iota^G/2$, for every $s \in [s_0, s_1]$. Hence by Lemma 4.0.23 with $X := [s_0, s_1]$ and the section a defined by $av := A(0, v)$ for $v \in T_q Q$ and $q \in Q$, there exists an equivariant diffeomorphism $g : [s_0, s_1] \times Q \to Q$ such that condition (49) is satisfied. We define the gauge transformation $\tilde{g} : P \to P$ by $\tilde{g}(s, q) := g(s)$, for every $s \in [s_0, s_1]$ and $q \in Q$, and we define $(A', u') := \tilde{g}^*(A, u)$. We fix $s \in [s_0 + 1, s_1 - 1]$. We have

$$\begin{aligned} \ell\big((u'(s, \cdot), a_0\big) &\leq \ell\big((A', u')(s, \cdot)\big) + \int_{S^1} \Big| \big(A'(s, \cdot) - a_0\big)_X \big(u'(s, \cdot)\big) \Big| (t) dt && (56) \\ &\leq \ell\big((A, u)(s, \cdot)\big) + C_1 \int_{S^1} \big|A'(s, \cdot) - a_0\big| dt. && (57) \end{aligned}$$

By inequality (80) we may estimate

$$\begin{aligned} \ell\big((A, u)(s, \cdot)\big) &\leq \int_{S^1} |d_A u|(s, t) dt \\ &\leq \int_{S^1} \sqrt{e^{\lambda}_{(A,u)}(s, t)} dt \\ &\leq 2\sqrt{C_3 E^{\lambda}(A, u)} \leq \frac{\delta}{2}. \end{aligned} \tag{58}$$

Here in the third step we used (53). On the other hand, by (49) and assumption (44), we have

$$\big|A'|_{\{s\times Q\}} - a_0\big|(t) = \bar{d}^G(\mathcal{C},\mathcal{C}_s) \le Ke^{-4\epsilon_0\min\{s-s_0,s_1-s\}}, \tag{59}$$

for every $t \in S^1$. Substituting this into (57) and combining with (58), we get

$$\ell\big((u'(s,\cdot),a_0\big) \le \delta.$$

This proves Claim 4.0.27. □

We choose a gauge transformation $g : P \to P$ as in Claim 4.0.27 and define $(A',u') := g(A,u)$. Then by (55) the condition (21) of Proposition 3.0.11 is satisfied with (A,u) replaced by (A',u'). It follows that the pairs $\big(u'(s_0,\cdot),a_0\big)$ and $\big(u'(s_1,\cdot),a_0\big)$ are admissible, and equality (22) holds with (A,u) replaced by (A',u'). We fix now $s \in \big[1,(s_1-s_0)/2-1\big]$. Using again (55) the condition (19) of Proposition 3.0.10 is satisfied with $x := u'(s+s_0,\cdot)$ and $a := A'|_{\{s+s_0\}\times Q}$. So applying this proposition with $p := 2$ and ϵ replaced by $e^{-4\epsilon_0 s}$, using $\min\{s,s_1-s_0-s\} = s$, we get

$$\begin{aligned}
\big|\mathcal{A}\big((A',u')(s+s_0,\cdot),a_0\big)\big| &\le C_2\big\|\mathrm{d}_{A'}(u'(s+s_0,\cdot))\big\|_2^2 + \left(\frac{e^{4\epsilon_0 s}}{4}+C_2\right)\big\|A'|_{\{s+s_0\}\times Q}-a_0\big\|_2^2 \\
&\quad + e^{-4\epsilon_0 s}\big\|u'^*P(\Phi)(s+s_0,\cdot)\big\|_2^2 \\
&\le C_2\int_{S^1}\big|\mathrm{d}_{A'}(u'(s+s_0,\cdot))\big|^2\,dt + \left(\frac{e^{4\epsilon_0 s}}{4}+C_2\right)K^2e^{-8\epsilon_0 s} \\
&\quad + \max_{x\in X}|\Phi(x)|^2e^{-4\epsilon_0 s} \\
&\le C_2\left(\int_{S^1}\big|\mathrm{d}_{A'}(u'(s+s_0,\cdot))\big|^2\,dt + e^{-4\epsilon_0 s}\right).
\end{aligned} \tag{60}$$

Here in the second step we used inequality (59). Similarly, we have

$$\big|\mathcal{A}\big((A',u')(s_1-s,\cdot),a_0\big)\big| \le C_2\left(\int_{S^1}\big|\mathrm{d}_{A'}(u'(s_1-s,\cdot))\big|^2\,dt + e^{-4\epsilon_0 s}\right). \tag{61}$$

It follows that

$$\begin{aligned}
f'(s) &= \frac{d}{ds}\left(\int_{[s_0+s,s_1-s]}\left(\int_0^1 e^{\lambda}_{A,u}dt + C_4e^{-4\epsilon_0 s}\right)ds\right)\\
&= -\int_0^1\Big(e^{\lambda}_{A,u}(s_0+s,t)+e^{\lambda}_{A,u}(s_1-s,t)\Big)dt - 4\epsilon_0C_4e^{-4\epsilon_0 s}\\
&\le -\int_0^1\Big(\big|d_{A'}(u'(s+s_0,\cdot))\big|^2+\big|d_{A'}(u'(s_1-s,\cdot))\big|^2\Big)dt - 4\epsilon_0C_4e^{-4\epsilon_0 s}\\
&\le -4\epsilon_0\Big(\big|\mathcal{A}\big((A',u')(s+s_0,\cdot),a_0\big)\big|+\big|\mathcal{A}\big((A',u')(s_1-s,\cdot),a_0\big)\big|\Big)\\
&\quad +2e^{-4\epsilon_0 s}-4\epsilon_0C_4e^{-4\epsilon_0 s}\\
&\le -2\epsilon_0\Big(E\big(A,u,\big[s+s_0,s_1-s\big]\times S^1\big)+C_4e^{-4\epsilon_0 s}\Big)\\
&\le -2\epsilon_0 f(s).
\end{aligned}$$

where in the fourth step we used (60), (61) and $4\epsilon_0 \le C_2^{-1}$, in the fifth step we used the energy action identity. This proves Claim 4.0.26. □

Claim 4.0.26 implies that on $\big[0,(s_1-s_0)/2\big]$,

$$\frac{d}{ds}\big(fe^{2cs}\big) = f'e^{2cs}+f2ce^{2cs}\le 0,$$

and hence

$$f(s)\le f(0)e^{-2cs}\le E(u,A)e^{-2cs}. \tag{62}$$

Combining this with (**??**) and (54), we get, for every $\hat{s}\in\big[s_0+1,(s_0+s_1)/2\big]$ and $\hat{t}\in S^1$,

$$|d_Au(\hat{s},\hat{t})|^2\le\frac{32}{\pi}f(\hat{s}-s_0-1/2)\le\frac{32e^c}{\pi}E(u,A)e^{-2c(\hat{s}-s_0)}.$$

Similarly, we get for every $\hat{s}\in\big[(s_0+s_1)/2,s_1-1\big]$ and $\hat{t}\in S^1$,

$$|d_Au(\hat{s},\hat{t})|^2\le\frac{32}{\pi}f(s_1-\hat{s}-1/2)\le\frac{32e^c}{\pi}E(u,A)e^{-2c(s_1-\hat{s})}.$$

Inequality (45) follows from this and (**??**). This proves Theorem 4.0.19. □

For $s\in[s_0,\infty)$ we denote by $i_s:\{s\}\times S^1\to\Sigma$ the inclusion.

Proposition 4.0.28. *Assume that X is closed. Let $\epsilon>0$ and $K>0$ be constants. Then there exists a constant C with the following property. Let $s_0\in\mathbb{R}$, $P\to\Sigma:[s_0,\infty)\times S^1$ be a principal G-bundle, $\lambda\in C^\infty\big([s_0,\infty)\times S^1,(0,\infty)\big)$, $p>2$ and (A,u) be a locally $W^{1,p}$-solution of the equation*

$$F_A+\lambda^2P(\Phi)\circ u\,ds\wedge dt=0. \tag{63}$$

Assume that

$$\lambda(s,t)\le Ke^{-\epsilon s},\quad\forall(s,t)\in\Sigma. \tag{64}$$

Then there exist a conjugacy class $\mathcal{C}_\infty \subseteq G$ and a constant C such that, denoting by $\mathcal{C}_s$ the holonomy of i_s^*A around S^1, we have

$$\bar{d}^G(\mathcal{C}_s, \mathcal{C}_\infty) \leq Ce^{-2\epsilon s}, \quad \forall s \in [s_0, \infty). \tag{65}$$

Proof of Proposition 4.0.28. We may assume without loss of generality that $P = [s_0, \infty) \times Q$. Gauge transforming the pair (A, u) we may assume that it is smooth and in temporal gauge, i.e. $A_{s,q}\big(\frac{\partial}{\partial s}\big) = 0$, for every $s \in [s_0, \infty)$, $q \in Q$. We fix a smooth connection a_0 on Q, and we define $\alpha_s := i_s^*A - a_0$. By (63), we have

$$ds \wedge \partial_s \alpha_s = F_A = -\lambda^2 Q(\Phi) \circ u\, ds \wedge dt. \tag{66}$$

Let $s_0 \leq s \leq s'$. Then for every $t \in S^1$, we have

$$\begin{aligned}
|\alpha_{s'}(t) - \alpha_s(t)| &= \int_{s_1}^{s'} |\partial_s \alpha_s(t)| ds \\
&\leq 2\epsilon_0 C_1 \int_s^{s'} e^{-2\epsilon s} ds \\
&\leq C_1\big(e^{-2\epsilon s} - e^{-2\epsilon s'}\big).
\end{aligned} \tag{67}$$

where $C_1 := \frac{K^2}{2\epsilon} \max_{x \in X} |\Phi(x)|$. Here in the second step we used (66). Hence the sequence α_ν, for $\nu \in \mathbb{N}$, is Cauchy in $C(S^1)$. So the same holds for the sequence of connections $a_0 + \alpha_\nu$. We denote by a_∞ its limit. Furthermore, we pick a point $q \in Q$, and denote by $h_\infty, h_s \in G$ the holonomies of a_∞ and i_s^*A respectively around S^1 starting at q. Moreover, we denote by $\mathcal{C}_\infty, \mathcal{C}_s \subseteq G$ the conjugacy classes of h_∞ and h_s respectively. To see that inequality (65) holds, we take the limit $s' \to \infty$ in (67), to obtain

$$\max_{S^1} |i_s^*A - a_\infty| \leq C_1 e^{-2\epsilon s}. \tag{68}$$

Hence i_s^*A converges to a_∞ in $C(S^1)$. Denoting by $\mathcal{C}_s$ the conjugacy class of the holonomy of i_s^*A, it follows that

$$\bar{d}^G(\mathcal{C}_s, \mathcal{C}_\infty) \leq d^G(h_\infty, h_s) \leq \int_0^1 |i_s^*A(t) - a_\infty(t)| dt \leq \frac{C_1}{2\epsilon} e^{-2\epsilon s}.$$

Here the second inequality follows from a standard argument, involving the derivative of the map $g_s : [0,1] \to G$ defined by $q_s g_s = q_\infty$, where $q_s, q_\infty : [0,1] \to Q$ are the i_s^*A- and a_∞-horizontal lifts of the projection $[0,1] \to S^1 \simeq \mathbb{R}/\mathbb{Z}$ starting at q. This proves Proposition 4.0.28. □

Corollary 4.0.29 (Decay for vortices on semi-infinite cylinders)**.** *Assume that X is closed. Then for every conjugacy class $\mathcal{C} \subseteq G$ there exists a constant $\epsilon_0 > 0$ with the following property. Let $s_0 \in \mathbb{R}$, $P \to \Sigma : [s_0, \infty) \times S^1$ be a principal G-bundle, $\lambda \in C^\infty\big([s_0, \infty) \times S^1, (0, \infty)\big)$, $p > 2$ and (A, u) be a locally $W^{1,p}$-solution of the vortex* *equations on P with area form $\omega_\Sigma := \lambda^2 ds \wedge dt$. Assume that*

$$K_0 := \sup_{(s,t)\in\Sigma} \lambda(s,t)e^{2\epsilon_0 s} < \infty, \tag{69}$$

$$K_1 := \sup_{(s,t)} \frac{9|d\lambda|^2}{\Delta(\lambda^2)} < \infty, \tag{70}$$

$$E(A,u) < \infty, \tag{71}$$

$$\bar{d}^G(\mathcal{C}_s, \mathcal{C}) \to 0, \quad \text{as } s \to \infty. \tag{72}$$

Then there exists a constant C such that

$$|d_A u|_0(s,t) \le Ce^{-\epsilon_0 s}, \quad \forall s \ge s_0 + 1,\, t \in S^1. \tag{73}$$

Proof of Corollary 4.0.29. Let $\mathcal{C} \subseteq G$ be a conjugacy class. We choose a constant ϵ_0 as in Theorem 4.0.19. Let $s_0 \in \mathbb{R}$, $P \to \Sigma : [s_0,\infty) \times S^1$ be a principal G-bundle, $\lambda \in C^\infty\big([s_0,\infty)\times S^1, (0,\infty)\big)$, $p > 2$ and (A,u) be a locally $W^{1,p}$-solution of the vortex equations on P with area form $\omega_\Sigma := \lambda^2 ds \wedge dt$. Assume that conditions (69,70,71,72) are satisfied. By (71) there exists a number $s_0' \ge s_0$ such that inequality (43) holds with s_0 replaced by s_0'. We may assume without loss of generality that $s_0' = 0$. We choose a constant $C_0 := C$ as in Proposition 4.0.28, corresponding to $\epsilon := 2\epsilon_0$ and $K := K_0$. We define

$$K := \max\left\{K_0, \sqrt{\max\{K_1,0\}}\, \max_X |\Phi|, C_0\right\}.$$

We fix a constant C as in the assertion of Theorem 4.0.19 corresponding to this K. Let $\hat{s} \ge 1$, $t \in S^1$. We define $s_1 := 2\hat{s}$. Then conditions (41) and (42) with s_0 replaced by 0 hold by assumptions (69) and (70) respectively. Furthermore, (44) with s_0 replaced by 0 follows from (65) and (72) (with C replaced by C_0). It follows that (45) holds with s_0 replaced by 0, and therefore $|d_A u|_0(\hat{s},t) \le Ce^{-\epsilon_0 \hat{s}}$. This proves (73) and completes the proof of Corollary 4.0.29. $\square$

5. Compactness

5.1. **Vortices with bounded first derivative.** The following extends compactness for bounded first derivative in Cieliebak-Gaio-Mundet-Salamon [2] to the case with cylindrical ends:

Theorem 5.1.1. *Let (A_α, u_α) be a sequence of vortices with bounded energy $E(A_\alpha,u_\alpha)$. If $\mathrm{d}u_\alpha$ is bounded in C^0 on compact sets, then there exists a smooth vortex (A_∞,u_∞) such that after gauge transformation and passing to a subsequence (A_α,u_α) converges to (A_∞,u_∞) uniformly in all derivatives on compact sets.*

Proof. Let (A_α,u_α) be a sequence as above. The vortex equation and bound on Φ give a pointwise bound on the curvature,

$$\|F_A\| < Ce^{\kappa_\Sigma r_\Sigma}.$$

This implies an L^2 bound for the curvature, see Proposition **??**. Uhlenbeck compactness implies that there is a subsequence (still denoted) A_α and a sequence of gauge transformations $g_\alpha \in \mathcal{G}(P)$ such that $g_\alpha \cdot A_\alpha$ converges weakly to a connection A_∞ in the local Sobolev topology $W_{m,p,\mathrm{loc}}$ and strongly in the C^0-topology (see Wehrheim [12, Theorem A']). By hypothesis the sequence $g_\alpha \cdot u_\alpha$ is bounded in $W_{m,p,loc}$ the theorems Alaoglu and Rellich, after passing to a subsequence, $g_\alpha \cdot u_\alpha$ converges weakly in $W_{m,p,loc}$ and strongly in C^0 to a section u_∞. Since F_{A_α} converges to F_{A_∞} and $\overline{\partial}_{J_{A_\alpha}} u_\alpha$ converges to $\overline{\partial}_{J_{A_\infty}} u_\infty$ both weakly in $W_{0,p,\delta}$, the pair (A_∞, u_∞) is a weak solution to the vortex equations in $W_{m,p,loc}$. By Proposition **??**, after gauge transformations, we can assume that (A_α, u_α) converges to (A_∞, u_∞) in $W_{m,p,\delta}$, and by Proposition **??** we may gauge transform so that (A_∞, u_∞) is actually smooth.

To show convergence in all derivatives on compact sets, we use the bootstrapping method of the proof of **??**, as follows. By Coulomb gauge, there is a sequence of gauge transformations $g_\alpha \in \mathcal{G}_{m+1,p,loc}$ such that $d^*_{A_\infty}(g_\alpha A_\alpha - A_\infty) = 0$. The sequence $g_\alpha A_\alpha$ has also L^2-bounded curvature and thus a subsequence, still denoted by $g_\alpha A_\alpha$, converges to A_∞ weakly in $W_{m,p,loc}$ and strongly in C^0. g_α is uniformly bounded in $W_{m+1,p,loc}$ and converges to an element $g \in \mathcal{G}_{m+1,p,loc}$ strongly in $W_{m,p,loc}$ and weakly in $W_{m+1,p,loc}$. By Proposition **??** a subsequence $g_\alpha \cdot u_\infty$ converges to $g \cdot u_\infty$ strongly in C^0 and weakly in $W_{m,p,\delta}$, and $g \cdot A_\infty = A_\infty$. Using equation (**??**), one sees that the convergence of A_α gives the convergence of almost complex structures J_{A_α} in $W_{m,p,loc}$. In particular, for every small open set U of compact support, this sequence is bounded uniformly, that is there is a constant c_0 such that, $\|J_{A_\alpha}\| \leq c_0$, for all α. Now, by [2, Lemma 3.3] we have a constant c, depending on c_0 and U so that, the sequence u_α of J_{A_α}-holomorphic sections is also bounded in $W_{m+1,p,loc}$, and by Proposition **??** we can assume it is also bounded in $W_{m+1,p,\delta}$. The sequences $a_\alpha := A_\alpha - A_\infty, u_\alpha$ are bounded in $W_{m,p,\delta}$, and they satisfy

$$d^*_{A_\infty}(a_\alpha) = 0, F_{A_\infty} + d_{A_\infty} a_\alpha + [a_\alpha, a_\alpha] + \omega_\Sigma (u_\alpha)^* P(\Phi) = 0.$$

Therefore, $d_{A_\infty}(a_\alpha)$ is bounded in $W_{m,p,\delta}$ as well as $d^*_{A_\infty}(a_\alpha)$, elliptic regularity shows that a_α is bounded in $W_{m+1,p,\delta}$. By passing to a subsequence if necessary, we can now assume that (A_α, u_α) converges in $W_{m+1,p,\delta}$. Continuing this process we get the convergence on compact sets in all derivatives up to gauge transformation. □

The case when the sequence $\mathrm{d}u_\alpha$ is not bounded on compact sets in the C^0 norm yields the existence of bubbles. More formally, we introduce the space of stable vortices which will be the proper compactification for the moduli space of vortices.

5.2. **Nodal Vortices.** Let Σ be a connected, oriented surface with n cylindrical ends.

Definition 5.2.1. A *combinatorial type* consists of a rooted tree Γ together with a

(a) partition of the vertices $\mathrm{Vert}(\Gamma) = \{0\} \cup \mathrm{Vert}(\Gamma)_C \cup \mathrm{Vert}(\Gamma)_s$ where vertex 0 is the *root vertex*. The vertices $\mathrm{Vert}(\Gamma)_C$ resp $\mathrm{Vert}(\Gamma)_s$ are *cylindrical* resp. *spherical*;
(b) a labelling of the cylindrical vertices by $\{1, \ldots, n\}$;
(c) a bijection from the set of semiinfinite edges of Γ to $\{1, \ldots, n\}$;

satisfying the condition that for each $i \in \{1, \ldots, n\}$, the vertices labelled i should form a linear string connected adjacent to the root vertex to the edge for the i-th marking.

In Figure **??**, the root resp. cylindrical resp. spherical vertices are black resp. grey resp. white. The three seminfinite edges are labeled $1, 2, 3$ and the single cylindrical vertex connects the root vertex with the third semininfinite edge.

The set of spherical vertices $\mathrm{Vert}(\Gamma)_s$ admits a partition

$$\mathrm{Vert}(\Gamma)_s = \mathrm{Vert}(\Gamma)_{\Sigma,0} \cup \mathrm{Vert}(\Gamma)_{\Sigma,\infty}$$

depending on whether they are connected to the root vertex by a path of spherical vertices or not. We call the second set the *spherical vertices at infinity.*

Definition 5.2.2. Let Γ be a combinatorial type. A *marked nodal curve with cylindrical ends of combinatorial type* Γ *with principal component* Σ is a nodal curve $\underline{\Sigma}$ consisting of components $\Sigma_0, \ldots, \Sigma_k$ and nodes $\{\{w_1^-, w_1^+\}, \ldots, \{w_m^-, w_m^+\}\}$ such that graph obtained by replacing components with vertices and nodes/markings with edges is the graph underlying Γ. A component is a *cylindrical* resp. *spherical* if the corresponding vertex is cylindrical resp. spherical.

Each cylindrical component Σ_i has two distinguished special points, given by the connecting points in the chain of cylinders connecting the root vertex with the marking, and hence a distinguished isomorphism $\Sigma_i \to \mathbb{P}^1$ mapping the two special points to $0, \infty$.

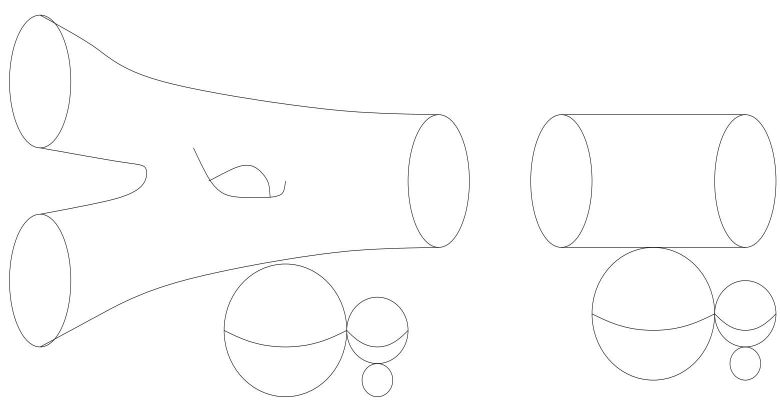

FIGURE 1. Example of a nodal curve with cylindrical ends

Definition 5.2.3. Let Γ be a combinatorial type, $P \to \Sigma$ a principal G-bundle, $J \in \mathcal{J}(P, X)$ and $\mathcal{H}(P, X)$ compatible almost complex structures and Hamiltonian perturbations respectively. A *nodal* (J, H)*-vortex* of combinatorial type Γ is a datum $(\underline{\Sigma}, \underline{A}, \underline{u}, \underline{z})$ consisting of

(a) a nodal curve $\underline{\Sigma}$ of combinatorial type Γ;
(b) a (J, H)-perturbed vortex on the principal component $\Sigma \cong \Sigma_0$.

(c) For each $i \in \mathrm{Vert}(\Gamma)_C$, a flat J-holomorphic vortex (A_i, u_i) on the cylinder $\Sigma_i - \{0, \infty\} \cong S^1 \times \mathbb{R}$ with holonomy given by the limiting holonomy of A_0 on the l-th end, where l is the label of i.

(d) For each $j \in \mathrm{Vert}(\Gamma)_s$ a $J_{\hat{z}_j}$-holomorphic sphere $v_j; \Sigma_j \to P(X)_{\hat{z}_j}$ in some fiber of $P(X)$;

satisfing matching conditions $\underline{u}(w_j^-) = \underline{u}(w_j^+)$ at the nodes $w_j^\pm, j = 1, \ldots, m$. We call the pair (Σ_j, v_j) a *spherical bubble* and the triple (Σ_i, A_i, u_i) a *cylinder bubble*. A bubble is *trivial* if the map is covariant constant on the bubble. A nodal vortex is *polystable* if each trivial bubble has at least three special points; that is, each cylinder bubble on which u is trivial is attached to at least one sphere bubble, and each sphere bubble has at least three special points. A nodal vortex is *stable* if it has finite automorphism group.

Definition 5.2.4. An *isomorphism* of polystable vortices $(\underline{A}_\alpha, \underline{u}_\alpha), \alpha = 0, 1$ consists of

(a) a gauge transformation over the principal component
(b) automorphisms of the domains of the bubbles preserving the special points

such that the action of the gauge transformation on the principal component and on the bundles over the fibers by evaluation at the special points, together with the automorphisms of domains, transforms $(\underline{A}_0, \underline{u}_0)$ to $(\underline{A}_1, \underline{u}_1)$.

Let $M_\Gamma(P, X, \underline{\mu})$ denote the moduli space of isomorphism classes of polystable vortices of combinatorial type Γ, and $M_\Gamma^{\mathrm{fr}}(P, X, \underline{\mu})$ the moduli space of isomorphism classes of framed polystable vortices. As in the case with smooth domain, $M_\Gamma^{\mathrm{fr}}(P, X, \underline{\mu})$ admits an evaluation map

$$\mathrm{ev}^{\mathrm{fr}} : M_\Gamma^{\mathrm{fr}}(P, X, \underline{\mu}) \to X^{n, \underline{\mu}} := \prod_{i=1}^n X^{\mu_i}.$$

If $M_\Gamma^{\mathrm{fr}}(P, X, \underline{\mu}) \to M_\Gamma(P, X, \underline{\mu})$ is a principal $G^n_{\underline{\mu}}$ bundle with a classifying map then combining this with the evaluation map defines

$$\mathrm{ev} : M_\Gamma(P, X, \underline{\mu}) \to X^{n,\underline{\mu}}_{G_{\underline{\mu}}}.$$

Let $\overline{M}(P, X, \underline{\mu})$ denote the union over combinatorial types

$$\overline{M}(P, X, \underline{\mu}) = \bigcup_\Gamma M_\Gamma(P, X, \underline{\mu}).$$

For any nodal curve $\underline{\Sigma}$, be denote by $Z_i \subset \Sigma_i$ the set of nodal points in Σ_i, that is, points at which other components of $\underline{\Sigma}$ are attached.

Definition 5.2.5. Suppose that (A_α, u_α) is a sequence of vortices on Σ and $(\underline{A}, \underline{u})$ is a polystable vortex. We say that (A_α, u_α) *Gromov converges* to $(\underline{A}, \underline{u})$ if there exist a sequence g_α of gauge transformations such that

(a) $g_\alpha A_\alpha$ converges uniformly to A on compact subsets of Σ_0;

(b) $u_{0,\alpha}$ converges to u_∞ uniformly on compact subsets of the complement of $Z_0 \subset \Sigma_0$ of $\underline{u}$;
(c) for every bubble component Σ_i of $\underline{\Sigma}$, there exists a sequence $\epsilon_{i,\alpha} \to \infty$ and maps $\phi_{i,\alpha} : \Sigma_i - B_{\epsilon_{i,\alpha}(w_i)} \to \Sigma$ such that $u_\alpha \circ \phi_{i,\alpha}$ converges uniformly on compact subsets of the complement of $Z_i \subset \Sigma_i$ to $u_{i,\infty}$.
(d) for any $w_j \in Z_j$, the energy lost

$$m(w_j) := \lim_{\epsilon \to 0} \lim_{\alpha \to \infty} E(u_\alpha \circ \phi_{i,\alpha}; B_\epsilon(z_j))$$

is equal to the sum of the energies on the components of $\underline{u}$ attached to w_j.
(e) for any $w_j \in Z_j$, $\phi_{\alpha,i}^{-1} \circ \phi_{\alpha,j}$ converges to w_j uniformly on compact sets in a neighborhood of w_i.
(f) if z_i is contained in Σ_j, then $z_i = \lim_{\alpha\to\infty} \phi_{j,\alpha}^{-1}(z_{i,\alpha})$.

A sequence $(A_\alpha, \underline{u}_\alpha)$ of polystable vortices Gromov-converges to a polystable vortex $(A, \underline{u})$ if there exists a sequence of contractions on the trees $\Gamma_\alpha \to \Gamma$, and properties similar to those above. The definition is similar to that in [7, Section 5.5] and omitted.

A subset C of $\overline{M}(P, X, \underline{\mu})$ is *Gromov closed* if any Gromov convergent sequence in C has limit point in C, and *Gromov open* if its complement is closed. This induces a topology in $\overline{M}(P, X, \underline{\mu})$.

Theorem 5.2.6. *$\overline{M}(P, X, \underline{\mu})$ is compact and Hausdorff. Furthermore, any convergent sequence is Gromov convergent.*

The proof takes up the rest of the section.

5.3. **Energy quantization.** The following is an energy quantization result for bubbles in the fibers of $P(X)$:

Lemma 5.3.1. *There exists a constant $\hbar > 0$ such that any non-constant holomorphic map $u : \mathbb{P}^1 \to P(X)$ having values in the fiber $P(X)$ has energy at least $\hbar$.*

Proof. By the standard result [7], since each fiber is equivariantly isomorphic (non-canonically) to X. □

Next we prove energy quantization of cylindrical bubbles with flat connections.

Proposition 5.3.2. *For any conjugacy class $\mathcal{C} \subset G$, there is a constant $\epsilon = \epsilon(\Sigma, \mathcal{C})$ such that any zero-area vortex (A, u) on the cylinder with holonomy in $\mathcal{C}$ and energy $E(A, u) < \epsilon(\Sigma, \mathcal{C})$ is trivial in the sense that u is covariant constant.*

We reduce this to energy quantization for holomorphic strips with Lagrangian boundary conditions.

Proposition 5.3.3. *Let (X, ω) be a compact symplectic manifold and $L_+, L_- \subset X$ compact Lagrangian submanifolds intersecting cleanly and $J \in \mathcal{J}(X)$ a tame almost complex structure. There exists a constant $\epsilon = \epsilon(L_+, L_-, J)$ such that if $u : \mathbb{R} \times [-1, 1] \to X$ is a J-holomorphic map satisfying the boundary conditions $u(\mathbb{R} \times \{\pm 1\}) \subset L_\pm$ and the energy estimate $E(u) < \epsilon$. then u is trivial.*

For the proof of Proposition 5.3.3 we need the following result. If X is a manifold and $Y \subset X$ is a submanifold, we denote by $N_X Y$ the normal bundle of Y in X.

Lemma 5.3.4. *Let X be a manifold, $i_\pm : Y_\pm \to X$ submanifolds intersecting cleanly, and $\omega \in \Omega(X)$ a closed differential form with $i_\pm^* \omega = 0$. Then there exists a neighborhood U of $Y_+ \cap Y_-$ and a form $\alpha \in \Omega(U)$ such that $\mathrm{d}\alpha = \omega|_U$ and $i_\pm|_{U \cap Y_\pm}^* \alpha = 0$.*

Proof. By the local model theorem for clean intersections [5, Proposition C.3.1], there exists a neighborhood U of $Y_+ \cap Y_-$ in X and a diffeomorphism φ of U to a neighborhood $\varphi(U)$ of the zero section in $N_x(Y_+ \cap Y_-)$ such that

$$\varphi(U \cap Y_\pm) = \varphi(U) \cap N_{Y_\pm}(Y_+ \cap Y_-).$$

In particular, $\varphi(U \cap Y_\pm)$ is a sub-bundle of $N_X(Y_+ \cap Y_-)$. Scalar multiplication on the fibers of $N_X(Y_+ \cap Y_-)$ defines a strong deformation retraction to $Y_+ \cap Y_-$,

$$\psi : [0, 1] \times U \to U, \quad (\lambda, x) \mapsto \varphi^{-1}(\lambda \varphi(x)).$$

Let $V_t \in \mathrm{Vect}(U)$ be the time-dependent vector field generating ψ,

$$V_t = \frac{d}{dt} \psi(x, t).$$

The Poincaré formula

$$\alpha = \int_0^1 \psi_t^* \iota(V_t) \omega \mathrm{d}t$$

produces the required primitive since

$$\begin{aligned} \mathrm{d}\alpha &= \int_0^1 \psi_t^* L_{V_t} \omega \mathrm{d}t \\ &= \int_0^1 \frac{d}{dt} \psi_t^* \omega \mathrm{d}t \\ &= \psi_1^* \omega - \psi_0^* \omega \\ &= \omega. \end{aligned}$$

The restriction of α to $Y_\pm$ is

$$\begin{aligned} i_\pm^* \alpha &= \int_0^1 \psi_t^* i_\pm^* \iota(V_t) \omega \mathrm{d}t \\ &= \int_0^1 \psi_t^* \iota(V_t) \iota_\pm^* \omega \mathrm{d}t \\ &= 0 \end{aligned}$$

since V_t is tangent to $Y_\pm$. The Lemma follows. □

Proof of Proposition 5.3.3. We set $Y_\pm := L_\pm$ and denote by $i_\pm : Y_\pm \to X$ the inclusions. Since $L_\pm$ is Lagrangian the hypotheses of Lemma 5.3.4 are satisfied. So let U and α be as in the conclusion of that lemma. Since X, L_+, L_- are compact, there exists a constant $\delta > 0$ such that if $x \in X$ then

$$\max(\mathrm{dist}(x, L_+), \mathrm{dist}(x, L_-)) < \delta \implies x \in U. \tag{74}$$

By the apriori lemma for holomorphic maps with Lagrangian boundary conditions [7, Lemma 4.3.1] there exists a constant $\epsilon_0 > 0$ such that if $u : \mathbb{R} \times [-1,1] \to X$ is a holomorphic map satisfying $u(\mathbb{R} \times \{\pm 1\}) \subseteq L_\pm$ and $E(u) \le \epsilon_0$ then

$$|du(z)|^2 \le \frac{8}{\pi} \int_{B_2(s,\pm 1) \cap \mathbb{R} \times [-1,1]} |du|^2, \tag{75}$$

for every $s \in \mathbb{R}$ and $z \in \bar{B}_1(s, \pm 1) \cap \mathbb{R} \times [-1,1]$. Let u be such a map. It follows that

$$\|\partial_t u(s,\cdot)\|_{L^1([-1,1])} \le 2\sqrt{\frac{8}{\pi} E\big(u, [s-2, s+2] \times [-1,1]\big)}, \tag{76}$$

for every $s \in \mathbb{R}$. We define

$$\epsilon := \min\Big\{\epsilon_0, \frac{\pi}{32}\delta^2\Big\}. \tag{77}$$

Assume that $E(u) \le \epsilon$. Fix $(s,t) \in \mathbb{R} \times [-1,1]$. Then

$$\begin{aligned} d\big(u(s,-1), u(s,t)\big) &\le \int_{-1}^{t} |\partial_t u(s,t')|\, dt' \\ &\le 2\sqrt{\frac{8}{\pi} E(u)} < \delta. \end{aligned} \tag{78}$$

Here in the second inequality we used (76) and in the last inequality we used (77). It follows that $\mathrm{dist}(u(s,t), L_-) < \delta$. Similarly, we have $\mathrm{dist}(u(s,t), L_+) < \delta$. By (74) we have

$$u(s,t) \in U. \tag{79}$$

Hence, denoting $E(s) := E\Big(u, \big((-\infty,-s) \cup (s,\infty)\big) \times [-1,1]\Big)$, we have for every $s \ge 2$

$$\begin{aligned} E(u) &= \int_{[-s,s]\times[-1,1]} u^*\omega + E(s) \\ &= \int_{\{s\}\times[-1,1]} u^*\alpha - \int_{\{-s\}\times[-1,1]} u^*\alpha + 0 + E(s) \\ &\le \Big(\|\partial_t u(s,\cdot)\|_{L^1([-1,1])} + \|\partial_t u(-s,\cdot)\|_{L^1([-1,1])}\Big)\|\alpha\|_{L^\infty(X)} + E(s). \end{aligned}$$

Here in the second step we used the fact $d\alpha = \omega|_U$, Stokes' theorem and the fact $i_\pm^*\alpha = 0$. By (76) the last expression tends to 0, as $s \to \infty$. It follows that $E(u) = 0$. This proves Proposition 5.3.3. □

Proof of Proposition 5.3.2. We choose a representative $g \in \mathcal{C}$ and define $\tilde{X} := X \times X$ with the symplectic form $\tilde{\omega} := -\omega \oplus \omega$ and the Lagrangian submanifolds

$$L_- := \{(x,x) \,|\, x \in X\}, \quad L_+ := \{(x, g^{-1}x) \,|\, x \in X\}.$$

Let (A,u) be a zero-area vortex on P; we view u as an equivariant map from P to X and A as a $\mathfrak{g}$-valued one-form on P. The pair (A,u) defines a J-holomorphic strip $\tilde{u}$ with boundary in $L_\pm$ as follows. Let $\pi : P \to \mathbb{R} \times S^1$ denote the projection and $\rho : \mathbb{R} \times [-1,1] \to \mathbb{R} \times S^1$ the be map $\rho(s,t) := (s, e^{\pi i t})$. Since A is flat and the conjugacy class of the holonomy around the circle $\{0\} \times S^1$ equals $\mathcal{C}$, there exists a map $f : \mathbb{R} \times [-1,1] \to P$ solving the equations

$$A(\partial_s f) = 0, \quad A(\partial_t f) = 0, \quad f(s,1) = f(s,-1)g, \quad \pi \circ f = \rho$$

for every $s \in \mathbb{R}$. Let $\tilde{X} = X \times X$ denote the Cartesian product equipped with the almost complex structure $\tilde{J} = -J \times J$. Let $\tilde{u} : \mathbb{R} \times [-1,1] \to X \times X$ denote the $\tilde{J}$-holomorphic map

$$\tilde{u}(s,t) = ((u \circ f)(s/2, -(t+1)/2), (u \circ f)(s/2, (t+1)/2)).$$

Then $\tilde{u}$ satisfies the boundary conditions $\tilde{u}(s, \pm 1) \in L_\pm$ and its energy is $E(\tilde{u}) = E(u,A)$. Since G is compact, X admits a G-invariant metric. Therefore, the hypothesis of Lemma 3.0.17 are satisfied and hence the intersection $L_- \cap L_+$ is clean. Therefore it follows by Proposition 5.3.3 that if $E(\tilde{u})$ has energy less than $\epsilon(L_-, L_+, \tilde{J})$ then $\tilde{u}$ is trivial, and hence u is covariant constant. This proves Proposition 5.3.2. □

5.4. **Removal of singularities.** We use the exponential decay results of the previous sections to prove a removal of singularities theorem for vortices (A,u) on the punctured disk. The result is not a full removal of singularities theorem: we assume that the connection has a C^0 extension. If we knew that the connection had a smooth extension, then removal of singularities would be a straightforward consequence of removal of singularities for pseudoholomorphic maps, viewing u as a pseudoholomorphic map to $P(X)$. However, Uhlenbeck compactness only provides a C^0 limit to the connection, so we have no control over the higher derivatives.

Proposition 5.4.1. *Suppose that (A,u) is a smooth finite energy vortex on the punctured disk $B - \{0\}$, and that A admits a C^0 extension over B. Then u admits a C^0 extension over B, and the pair (A,u) is gauge equivalent to a smooth vortex on B.*

Sketch of proof. Removal of singularities for pseudoholomorphic maps is [7, Lemma 4.5.1]. The proof for vortices is similar: one first shows that u admits a $W^{1,p}$ extension for $p > 2$ with $2 - p$ sufficiently small. Elliptic regularity then implies that the pair (A,u) is gauge equivalent to a smooth vortex.

Suppose that the mean value inequality of Proposition **??** holds for all balls of energy at most C. Choose r_0 so that $E(A,u;B_{2r_0}) < C$. For $0 < r \leq r_0$ let

$$\epsilon(r) = E(A,u;B_r)$$

denote the energy of the restriction of (A,u) to B_r. Then ϵ extends to a continuous function on $[0,r_0]$ with $\epsilon(0)=0$. By the mean value inequality **??**

$$|\nabla_A u(re^{i\theta})|^2 \leq C\epsilon(2r).$$

This implies that the length of $\gamma_r(\theta)=u(re^{i\theta})$ satisfies

$$\ell(\gamma_r) \leq \sqrt{C\epsilon(2r)}$$

which goes to zero as $r\to 0$. From the isoperimetric inequality (**??**) (for trivial twisting $g_0=1$) we obtain

$$E(A,u;B_r) \leq c\ell(\gamma_r)^2.$$

Hence

$$\begin{aligned}
\epsilon(r) &\leq c\ell(\gamma_r)^2 \\
&\leq \pi c r^2 \int_{S^1} |\nabla_A u(re^{i\theta})|^2 \\
&= 2\pi c r \frac{d}{dr}\epsilon(r).
\end{aligned}$$

Integrating from r to r_1 gives for $\mu = 1/2\pi c$

$$\epsilon(r) \leq c_1 r^{\mu}.$$

So

$$|\mathrm{d}u(\rho e^{i\theta})|^2 \leq C\rho^{-2}\epsilon(2\rho) \leq C\rho^{2\mu-2}.$$

Hence for p with $2<p<2/(1-\mu)$

$$\int_{B_r} |\nabla_A u|^p \leq C\int_0^r \rho^{1-p(1-\mu)}\mathrm{d}\rho < \infty.$$

It follows as in [7] that u lies in $W^{1,p}(B_r)$. Now elliptic regularity Proposition **??** implies that (A,u) is gauge equivalent to a smooth solution. □

5.5. **Gromov compactness for manifolds with cylindrical ends.** Let X be a manifold. A *cylindrical end* of X is a proper embedding $\phi:[0,\infty)\times Y\to X$, where Y is a closed manifold of dimension $\dim X-1$. We say that X has cylindrical ends if there exists a collection $(Y_i,\phi_i)_{i\in I}$ of cylindrical ends of X such that $X\setminus\bigcup_{i\in I}\phi_i\big((0,\infty)\times Y_i\big)$ is compact. Let X be a manifold with cylindrical ends. A *collection of cylindrical ends* for a principal bundle $P\to X$ is a collection $Q_i\to Y_i$ of principal G-bundles on the manifolds Y_i, and isomorphisms $\phi_i^*P\to\pi_i^*Q_i$ where $\pi_i:[0,\infty)\times Y_i\to Y_i$. Any principal bundle over X admits cylindrical ends, by parallel transport using a connection. Let $P\to X$ be a principal G-bundle equipped with cylindrical ends and F a G-manifold. The associated fiber bundle $P(F)=(P\times F)/G$ is then a manifold with cylindrical ends modelled on $Q_i(F)$ on the i-th cylindrical end.

An almost complex structure J on a manifold with cylindrical ends X is *asymptotically constant* if for each end, there exists an almost complex structure J_i on $\mathbb{R}\times Y_i$ invariant under translations such that $\psi^*J(\cdot+s,\cdot)$ converges to J_i as $s\to\infty$ in $W^{loc}_{1,p}$. Let J be an asymptotically constant almost complex structure on X. A sequence J_ν of almost complex structure on X converges to J if $\psi^* J_\nu(s,t)$ converges to J_i as $s \to \infty$ uniformly in ν.

The following terminology is taken from symplectic field theory [**?**]. Our situation is somewhat easier since the curve Σ is fixed. Let $\pi_l : Y_l \times \mathbb{R} \to Y_l$ denote the projection on the first factor.

Definition 5.5.1. A *holomorphic building* in X consists of a nodal curve with cylindrical ends $\underline{\Sigma}$ consisting of a principal component Σ_0 and sphere or cylinder bubbles $\Sigma_1, \dots, \Sigma_k$, a holomorphic map $u_0 : \Sigma_0 \to X$ and for each bubble a holomorphic map from Σ_i to either X or to one of the cylindrical ends $Y_l \times \mathbb{R}$, satisfying matching conditions at the nodes $u_{i^+(j)}(w_j^+) = u_{i^-(j)}(w_j^-)$ if the node maps to X and

$$\pi_l(\lim_{z \to w_j^+} u_{i^+(j)}(z)) = \pi_l(\lim_{z \to w_j^-} u_{i^-(j)}(z))$$

for the nodes mapping to infinity on the cylindrical end. A holomorphic building is *stable* if each sphere or cylinder bubble on which $\underline{u}$ is trivial has at least three nodal points.

An *admissible annulus* in Σ is an open holomorphic embedding $\psi : (s_0, s_1) \times S^1 \to \Sigma$ such that one of the following conditions is satisfied:

(a) $s_0 \geq 0$ and there exists a cylindrical end $\phi : (0, \infty) \times S^1 \to \Sigma$ such that ψ is the composition of ϕ with the inclusion $(s_0, s_1) \times S^1 \to (0, \infty) \times S^1$; or
(b) There exists a ball $\phi : B_r \subset \mathbb{C}$ and a holomorphic embedding $B_r \to \Sigma$ such that ψ is the composition of ϕ with the exponential map $(s_0, s_1) \times S^1 \to \mathbb{C}$, $(r, \theta) \mapsto e^{r + i\theta}$.

Theorem 5.5.2. *Let (Σ, j) be a Riemann surface without boundary, ω_Σ be a j-compatible area form on Σ, X be a manifold without boundary and with finitely many cylindrical ends, $p > 2$, for $\nu \in \mathbb{N}$ let ω_ν be a symplectic form on X, J_ν be an ω_ν-compatible almost complex structure on X, both of class locally $W^{1,p}$, and let $u_\nu \in W^{1,p}_{\mathrm{loc}}(\Sigma, X)$ be a J_ν-holomorphic map. Assume that J_ν converges to some J_0, and ω_ν converges to some ω_0, both weakly in $W^{1,p}_{\mathrm{loc}}$, for every compact subset $Q \subseteq \Sigma$ there exists a compact subset $K \subseteq X$ such that $u_\nu(Q) \subseteq K$, for every ν,*

$$\sup_\nu E_{\omega_\nu, J_\nu}(u_\nu, \Sigma) < \infty, \tag{80}$$

(81)

and there exist constants $C, \epsilon > 0$ such that for each admissible annulus $\psi : (s_0, s_1) \times S^1 \to \Sigma$ satisfying $E^{\omega_\nu, J_\nu}(\psi^* u_\nu) < \epsilon$, we have

$$|du_\nu|_{\omega_\nu, J_\nu} \leq C e^{-\epsilon \min(s - s_0, s_1 - s)}$$

then there exists a subsequence of u_ν that converges to a stable holomorphic building $\underline{u} : \underline{\Sigma} \to X$.

5.6. **Proof of compactness.**

Suppose that (A_α, u_α) is a sequence of vortices.

Definition 5.6.1. $z \in \Sigma$ is a *bubble point* for the sequence (A_α, u_α) if there exists a convergent sequence $z_\alpha \to z$ such that $\mathrm{d}_{A_\alpha} u_\alpha(z_\alpha) \to \infty$.

Proposition 5.6.2. *If (A_α, u_α) is a sequence of vortices whose energy is bounded by C, then there is a finite set of bubbling points $Z \in \Sigma$, and a vortex (A_∞, u_∞) on Σ such that a subsequence still denoted (A_α, u_α), converges after gauge transformations to (A_∞, u_∞) on compact sets of $\Sigma \backslash Z$ in all derivatives.*

Proof. To show this, for a bubbling point s, $\lim_{z_\alpha \to s} |\mathrm{d}_{A_\alpha} u_\alpha(z_\alpha)| = \infty$. Let $\epsilon > 0$ small enough so that a neighborhood of s is consider as an open set in $\mathbb{C}$. By Hofer's lemma [7, 4.6.4] to the function $z \mapsto |d_{A_\alpha} u(z)|$ for $|z - s| < \epsilon$, the points z_α and the constants $\delta_\alpha := |d_{A_\alpha} u_\alpha(z_\alpha)|^{-1/2}$, there exist sequences $\zeta_\alpha \in \Sigma$, $\epsilon_\alpha > 0$ such that

$$\zeta_\alpha \to s; \quad \sup_{|z-\zeta_\alpha|<\epsilon_\alpha} |d_{A_\alpha} u_\alpha| \leq 2c_\alpha; \quad \epsilon_\alpha \to 0; \quad \epsilon_\alpha c_\alpha \to \infty,$$

where $c_\alpha := |d_{A_\alpha} u_\alpha(\zeta_\alpha)|$. Let ψ_α be the sequence defined on the ball $B_{\epsilon_\alpha c_\alpha}(0)$ given by $\psi_\alpha(z) := (\zeta_\alpha + z/c_\alpha)$. This sequence converges to s uniformly in compact sets. The rescaled sequence

$$v_\alpha(z) := u_\alpha(\psi_\alpha(z))$$

has uniformly bounded first derivative on compact sets, on arbitrarily large domains, since the sequence $\epsilon_\alpha c_\alpha \to \infty$. The maps v_α are pseudoholomorphic with respect to the almost complex structure determined by the re-scaled connections $c_\alpha^{-1} A_\alpha$, which have uniformly bounded curvature. Using Theorem 5.1.1 we get, after gauge transformations, that the sequence $(c_\alpha^{-1} A_\alpha, v_\alpha)$ converges uniformly to a pair (A_0, v) on compact sets, in all derivatives. Note that the limit A_0 is necessarily the trivial connection, since $c_\alpha \to \infty$, and that the map v necessarily lies on the fibre s,

$$v : \mathbb{C} \to P(X)_s$$

since standard removal of singularities for finite energy maps implies that v extends to a smooth J-holomorphic map $v : \mathbb{P}^1 \to P(X)_s$, where J is the given almost complex structure on X, and v is non constant.

By energy quantization, sphere bubbles can develop at most at finitely many points Z. On the complement $\Sigma - Z$, $d_\alpha u_\alpha$ is uniformly bounded in compact sets and (A_α, u_α) has bounded energy, then by Theorem 5.1.1 A_α converges to a connection A_∞ on compact subsets of Σ and u_α converges to a section u_∞ on compact subsets of $\Sigma - Z$ in all derivatives, so that (A_∞, u_∞) is a solution to the vortex equations on $\Sigma - Z$. Since u_∞ is a finite energy pseudoholomorphic map from Σ to $P(X)$, removal of singularities implies that it extends to all of Σ. (A_∞, u_∞) is the *principal component* of the limiting sequence. □

Let $s \in \Sigma$ be a bubbling point, and v_α the sequence obtained by re-scaling as in the proof of Proposition 5.6.2. Fix a trivialization of P in a neighborhood of s. Let

$$m(s) := \lim_{\epsilon \to 0} \lim_{\alpha} (u_\alpha; B_\epsilon(s))$$

be the energy of the sequence being captured at s. By choosing a subsequence, this limit exists since the energy is uniformly bounded. Since bubbling occurs near s, we can restrict the sequence u_α to the ball $B_R(s)$ for $R > 0$. Moreover, let $\zeta'_\alpha \in B_R(s)$

be the point where the function $d_{A_\alpha} u_\alpha$ attains its supremum on $B_R(s)$. The section $\tilde{u}_\alpha := u_\alpha(z + \zeta'_\alpha)$ is such that attain its sup at $z = 0$ and thus the sequence $\tilde{u}_\alpha$ has a bubbling point at $z = 0$, since the sequence $\zeta'_\alpha \to s$.

Definition 5.6.3. The sequence obtained by *soft rescaling* is the sequence $v_\alpha(z) = u_\alpha(\epsilon_\alpha z)$ where the rescaling constants ϵ_α are defined so that

$$E(\tilde{u}_\alpha; B_{\epsilon_\alpha}(0)) = m(s) - h/2, \tag{82}$$

where $h < \min\{\hbar/2, c_2/2\}$ where $\hbar$ is the energy quantization constant and c_2 is the constant guaranteeing exponential decay in Theorem **??**.

For any $\epsilon > \epsilon_\alpha$ the energy of $\tilde{u}_\alpha$ on the annulus $B_\epsilon(0) \backslash B_{\epsilon_\alpha}(0)$ is at the most $\delta/2$ and then there is not enough energy to form another bubble in this annulus. Let $\psi_\alpha(z) := \zeta'_\alpha + \epsilon_\alpha z$ and let $(A'_\alpha(z), v_\alpha(z)) := (A_\alpha(\psi_\alpha), u_\alpha(\psi_\alpha))$ denote the rescaled sequence defined on the ball $B_{R/\epsilon_\alpha}(0)$.

Proposition 5.6.4 (Bubbles connect)**.** *Consider the sequence v_α constructed by soft rescaling above.*

(a) *There is a finite energy pseudoholomorphic map $v : \mathbb{C} \to P(X)_s$ and a finite set of points $Z_1 \subset \mathbb{C}$ so that the sequence v'_α converges uniformly in all derivatives on $\mathbb{C} \backslash Z_1$ to v. By removal of singularities v extends to $\mathbb{P}^1$.*
(b) *Let $m_j := \lim_{\epsilon \to 0} \lim_\alpha E(v_\alpha; B_\epsilon(s_j))$ is the energy being captured by the bubbling point $s_j \in Z_1$, then*
$$m_0 := E(v) + \sum_{s_j \in Z_1} m_j.$$
Therefore, there is no other possible bubbles forming at s, only at the points $s_j \in Z_1$.
(c) *We have $u(z) = v(\infty)$ in $P(X)_z$.*

Proof. This sequence by construction has uniformly bounded energy. By the previous discussion, there exists a finite set $Z_1 \subset \mathbb{C}$, and a vortex (A'_0, v) such that a subsequence still denoted (A'_α, v_α) converges uniformly on compact sets in all derivatives to (A'_0, v). Since $\epsilon_\alpha \to 0$, A' is necessarily the trivial connection. Also note that the function $|dv(z)|$ has its maximum at 0, thus $0 \in Z_1$. v has finite energy, and since $\delta_\alpha \to 0$, it is defined on arbitrarily big subsets of $\mathbb{C}$. Removal of singularities shows that it extends to a J-holomorphic map $v : \mathbb{P}^1 \to P(X)_s$, with J exactly the almost complex structure on X, since the connection A' is trivial. Now, recall that by Step 1, a subsequence of (A_α, u_α) converges in all derivatives on compact sets of $\Sigma \backslash Z$ to the principal component vortex (A_∞, u_∞). The pair (A_0, v) is the *first bubble* that appears attached to the principal component.

The rest of the proposition is as in the proof of [7, 4.7.1], using that the exponential decay for the energy on annuli near s for the sequence u_α of J_{A_α}-holomorphic curves proved in Proposition **??**.

Let $m(s)$ be the energy lost at the point s. Note that we do not need to distinguish the Yang-Mills-Higgs energy with the twisted energy $E_{A_\alpha}(u_\alpha; B_\epsilon(s))$ here, since the energy of the connection on $B_\epsilon(s)$ approaches zero. That is, we also have

$$m(s) = \lim_{\epsilon\to 0}\lim_{\alpha\to\infty} E_{A_\alpha}(u_\alpha; B_\epsilon(s)).$$

Note that $\lim_{R\to\infty}\lim E_{A_\alpha}(u_\alpha; B_{R\epsilon_\alpha}) = m(s)$, since otherwise it would exist a subsequence still denoted by u_α and a constant $\rho > 0$ such that for $R \geq 1$,

$$\lim_{\alpha} E_{A_\alpha}(u_\alpha; B_{R\epsilon_\alpha}) \leq m(s) - \rho.$$

Thus for $R > 1$, the energy in the annuli satisfies $\lim_\alpha E_{A_\alpha}(u_\alpha; A(\epsilon_\alpha, R\epsilon_\alpha)) \leq c_2/2 - \rho$, which is a contradiction.

To finish the proof, note that $E_{A_\alpha}(u_\alpha; B_{\epsilon_\alpha}(s)) = E_{A_\alpha}(v_\alpha; B_1(s)) = m_0 - h/2 \geq m_0 - \hbar/2$, and both sequences u_α, v_α capture energy $m(s)$ at s. This shows that $Z_1 \subset B_1(s)$, that is all bubbling points of v_α are in the unit ball of s. Then, for all balls $B_\epsilon(s) \subset B_1(s)$ we have

$$\begin{aligned}
m(s) &= \lim_{R\to\infty}\lim_{\alpha} E_{A_\alpha}(v_\alpha; B_R(s)) \\
&= \lim_{R\to\infty}\lim_{\alpha} E_{A_\alpha}(v_\alpha; B_R(s)\backslash B_\epsilon(s)) + \lim_{\alpha} E_{A_\alpha}(v_\alpha; B_\epsilon(s)) \\
&= \lim_{R\to\infty} E(v; B_R(s)\backslash B_\epsilon(s)) + \lim_{\alpha} E_{A_\alpha}(v_\alpha; B_\epsilon(s)) \\
&= E(v; \mathbb{C}\backslash B_\epsilon(s)) + \lim_{\rho\to 0}\lim_{\alpha} E_{A_\alpha}(v_\alpha; B_\epsilon(s)\backslash \cup_{j\in Z_1} B_\rho(s_j)) + \sum_{s_j\in Z_1} m(s_j) \\
&= E(v) + \sum_{s_j\in Z_1} m(s_j).
\end{aligned}$$

By definition, there exists a sequence $\kappa_\alpha \to 0$ such that

$$\lim_{\alpha\to\infty} E_{A_\alpha}(u_\alpha; B_{\kappa_\alpha}(s)) \to m(s).$$

$[\log(\delta_\alpha), \log(\kappa_\alpha)] \times S^1$. By the uniform mean value inequality (**??**), on the subset $[\log(\delta_\alpha)+1, \log(\kappa_\alpha)-1] \times S^1$ the twisted derivatives $\mathrm{d}_{A_\alpha} u_\alpha$ are uniformly bounded. The exponential decay lemma **??** shows exponential decay of the energy on this region. Recall from [7, p. 103] that since there is not enough energy for bubbling, the energy on the outer region of the annulus must approach zero. Hence the energy density on the annulus is controlled by the energy on the inner region.

Since the connections A_α are already in Coulomb gauge with respect to the trivial connection, exponential decay of the distance. We have

$$u_\infty(z) = \lim_{\alpha\to\infty}(u_\alpha(\kappa_\alpha)), \quad v_\infty(\infty) = \lim_{\alpha\to\infty}(u_\alpha(\delta_\alpha))$$

since there is not enough energy on the annulus $A(\delta_\alpha, \kappa_\alpha)$ for further bubbling.

Hence

$$\operatorname{dist}(v_\infty(\infty), u_\infty(0)) \le \lim_{\alpha\to\infty} \operatorname{dist}(u_\alpha(\kappa_\alpha), u_\alpha(\delta_\alpha)) = 0$$

by exponential decay. □

We can now inductively construct the spherical components of a *stable* holomorphic section on the fibre $P(X)_s$ by applying the previous step to all points $s_j \in Z_1$. All bubble maps are holomorphic with respect to the trivial connection, and thus one can just think of lines in X. This process stops since the energy m_0 is finite. The case when Z has more than one element is left to the reader.

It remains to construct the bubble trees attached to the cylindrical ends. If $\sup_\alpha |\mathrm{d}_{A_\alpha}(u_\alpha(z_\alpha))|$ is unbounded for some sequence $z_\alpha \to z_i$ going to infinity on the i-th cylindrical end, there are two possibilities that can happen. First, on the cylindrical ends there is also translational symmetry, which means that other vortices can form at infinity on the cylindrical ends. Second, spherical bubbles on these cylindrical bubbles can also form. For any time $s \ge 0$, let $\tau_s : S^1 \times (0,\infty) \to S^1 \times (0,\infty)$ denote translation by s.

Suppose that there is a sequence of points z_α on the cylindrical end such that the r-coordinate of z_α goes to infinity and $\mathrm{d}_{A_\alpha} u(z_\alpha)$ is bounded from above and below. Thus, there must be a sequence of numbers r_α,a vortex (B, v) on the trivial bundle over the cylinder $S^1 \times \mathbb{R}$, satisfying the flat limit of the vortex equations

$$F_B = 0, \quad \overline{\partial}_{J_B} v = 0, \tag{83}$$

and a finite set Z so that $\tau_{r_\alpha}(A_\alpha, u_\alpha)$ converges on compact sets on the complement $\Sigma\backslash Z$ to (B, v) in all derivatives. Note that after gauge transformation any connection on the cylinder is in temporal gauge, if flat it is then of the form $A = \mathrm{d} + a\mathrm{d}\theta$, for a constant element $a \in \mathfrak{g}$, and d the trivial connection. Then, the holonomy at infinity of the limit vortex agrees with the holonomy at infinity of the sequence. By energy quantization for g-twisted pseudoholomorphic cylinders Theorem **??**, only finitely many cylindrical bubbles can occur.

To capture the first bubble, let $m_0 := \lim_T \lim_\alpha (A_\alpha, u_\alpha; (T;\infty) \times S^1)$ be the energy of the sequence that dissipates at infinity. Let h be a constant smaller than $\min\{\hbar, k\}$, where k is the constant in Theorem **??**. Choose r_α so that the energy $E(u_\alpha; (r_\alpha, \infty) \times S^1) = m_0 - h/2$, by gauge transforming the pairs (A_α, u_α) if necessary and choosing a subsequence, we can assume that r_α is bigger than the constant r_0 of Theorem **??**, so that the covariant derivative $\nabla_A u$ and the energy $E(u)$ have exponential decay on the cylindrical end. Let $(B_\alpha, v_\alpha) := \tau_{r_\alpha}(A_\alpha, u_\alpha)$ be the rescaled sequence. This sequence has bounded energy vortices. By using Step 1, there is a finite set $Z \subset S^1 \times \mathbb{R}$ and a flat vortex (B, v) on the cylinder such that (B_α, v_α) converges in $\mathbb{R} \times S^1\backslash Z$ in compact sets with all derivatives. Moreover, the choice of rescaling ensures that no further bubbling for the original sequence can happen at infinity in the i-th cylindrical end. Exponential decay of u at infinity, as well as exponential decay for flat vortices as stated in Proposition **??** ensures the existence of limits which by construction connect: $u(\infty) = v(-\infty)$.

We now apply this process inductively, after capturing the first bubble, the limiting sequence is as follows. For $j = 1, \dots, M$ there are sequences $\{r^j_\alpha\}$ of positive numbers such that $r^j_\alpha \to \infty$ as $\alpha \to \infty$, flat vortices (A_j, v_j) on the cylinder $\mathbb{R} \times S^1$ and finite sets of points $Z_j \subset \mathbb{R} \times S^1$ so that $\tau_{r_\alpha}(A_\alpha, u_\alpha)$ converges to (B_j, v_j) on $\mathbb{R} \times S^1 \backslash Z_j$ on compact sets and there is spherical fibre bubbling occurring on points $s \in Z_j$. Now apply the arguments used in Case 1 to build the bubble tree on the fibres $P(X)_s, s \in Z_j$. The cylindrical bubbles also have limits at infinity, since the exponential decay for finite energy flat vortices Proposition **??** and by construction they connect: $v_j(\infty) = v_{j+1}(-\infty)$. This finishes the proof of the Theorem 5.2.6.

5.7. **Local distance functions.** To show that the topology is Hausdorff we need to describe Gromov convergence of stable vortices in terms of an auxiliary function called the distance function. Our function is the same as in [7, p. 134] with an added term including the connection.

Definition 5.7.1. The *local distance function* ρ_ϵ for $\epsilon > 0$ is

$$\rho_\epsilon((\underline{A}, \underline{u}), (\underline{A}', \underline{u}')) = \inf_{f:T \to T'} \inf_{g \in \mathcal{G}(P)} \inf_{\underline{\phi}} \rho_\epsilon((\underline{A}, \underline{u}), g(\underline{A}', \underline{u}'); f, \underline{\phi}),$$

where

$$\begin{aligned} \rho_\epsilon((\underline{A}, \underline{u}), (\underline{A}', \underline{u}'); f, \underline{\phi}) := & \|A' - A\|_{L^2} \\ & + \sup_j |E(A, \underline{u}; B_\epsilon(w_j)) - E(A', \underline{u}'; \underline{\phi}(B_\epsilon(w_j)))| \\ & + \sup_{i \in \Gamma} \sup_{z \notin B_\epsilon(w_i)} d(u'_{f(i)} \phi_i, u_i) \\ & + \sup_{i \neq j, f(i) = f(j)} \sup_{z \notin B_\epsilon(w_j)} d(\phi^{-1}_{i^+(j)} \phi_{i^-(j)}, w_j) \\ & + \sup_{f(i) \neq f(j)} d(\phi_j^{-1}(z_{f(i)f(j)}), w_i) \\ & + \sup_{i \in T, 1 \leq j \leq n} d(\phi_i^{-1}(z'_{f(i)j}), w_i) \end{aligned}$$

depends on the contraction $f : \Gamma \to \Gamma'$ such that map the nodes $i \to i'$ and

$$\underline{\phi} = (\phi_1, \dots, \phi_n), \quad \phi_i : \Sigma_i - B_\epsilon(Z_i) \to \Sigma'_{f(i)}$$

is holomorphic isomorphism of $\Sigma_i - B_\epsilon(Z_i)$ onto its image. We set $\rho_\epsilon = \infty$ if there are no contractions f.

Lemma 5.7.2. *For ϵ sufficiently small, $(\underline{A}_\alpha, \underline{u}_\alpha)$ Gromov converges to $\underline{A}, \underline{u}$, if and only if $\rho_\epsilon((\underline{A}_\alpha, \underline{u}_\alpha), (\underline{A}, \underline{u}))$ converges to zero.*

Proof. This is the vortex version of [7, Lemma 5.5.8]. The forward direction is immediate from the definition of Gromov convergence. The reverse implication holds for any ϵ sufficiently small so that the energy of $(A, \underline{u})$ on B_ϵ is less than $\hbar/2$ where $\hbar$ is the energy quantization constant of the previous section. Since $\rho_\epsilon \to 0$, $(\underline{A}_\alpha, \underline{u}_\alpha)$ converges to $(A, \underline{u})$ uniformly on compact subsets of the complements of the balls $B_\epsilon(w_j)$. Since there is no additional bubbling, $(\underline{A}_\alpha, \underline{u}_\alpha)$ converges to some limit $(\underline{A}_\infty, \underline{u}_\infty)$ with the same combinatorial type as $(\underline{A}, \underline{u})$ and $(\underline{A}_\infty, \underline{u}_\infty) = (\underline{A}, \underline{u})$ on the complement of the balls $B_\epsilon(w_j)$. By unique continuation for pseudoholomorphic maps, $\underline{u}_\infty = \underline{u}$ everywhere on $\underline{\Sigma}$, which completes the proof. □

Proposition 5.7.3. *The Gromov open sets form a topology for which any convergent sequence is Gromov convergent. Furthermore, any convergent sequence has a unique limit.*

Proof. By [7, Lemma 5.6.5] it suffices to show that for all $(\underline{A}, \underline{u})$ there is an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, the function ρ_ϵ satisfies the following:

(a) $\rho_\epsilon((\underline{A}, \underline{u}), (\underline{A}', \underline{u}')) = 0$ if and only if $(\underline{A}, \underline{u}) = (\underline{A}', \underline{u}')$.
(b) $(\underline{A}_\alpha, \underline{u}_\alpha)$ converges to $(\underline{A}, \underline{u})$ if and only if $\rho_\epsilon((\underline{A}_\alpha, \underline{u}_\alpha), (\underline{A}, \underline{u}))$ converges to 0.
(c) Suppose that $(\underline{A}_\alpha, \underline{u}_\alpha)$ converges to $(\underline{A}, \underline{u})$. Then $\limsup_\alpha \rho_\epsilon((A_\alpha, \underline{u}_\alpha), (A', \underline{u}')) \leq \rho_\epsilon((\underline{A}, \underline{u}), (A', \underline{u}'))$.

(a) Suppose $\rho_\epsilon((\underline{A}, \underline{u}), (\underline{A}', \underline{u}')) = 0$. Then after gauge transformation and reparametrization $\underline{A} = \underline{A}'$, and $\underline{u} = \underline{u}'$. (b) and (c) follow from Lemma 5.7.2. □

References

[1] U. Bhosle and A. Ramanathan. Moduli of parabolic G-bundles on curves. *Math. Z.*, 202(2):161–180, 1989.
[2] Kai Cieliebak, A. Rita Gaio, Ignasi Mundet i Riera, and Dietmar A. Salamon. The symplectic vortex equations and invariants of Hamiltonian group actions. *J. Symplectic Geom.*, 1(3):543–645, 2002.
[3] M. Furuta and B. Steer. Seifert fibred homology 3-spheres and the Yang-Mills equations on Riemann surfaces with marked points. *Adv. Math.*, 96(1):38–102, 1992.
[4] Ana Rita Pires Gaio and Dietmar A. Salamon. Gromov-Witten invariants of symplectic quotients and adiabatic limits. *J. Symplectic Geom.*, 3(1):55–159, 2005.
[5] L. Hörmander. *The analysis of linear partial differential operators. III.* Springer-Verlag, Berlin, 1994. Pseudo-differential operators, Corrected reprint of the 1985 original.
[6] Ignasi Mundet i Riera and Gang Tian. A compactification of the moduli space of twisted holomorphic maps. arXiv:math.SG/0404407.
[7] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.
[8] V. B. Mehta and C. S. Seshadri. Moduli of vector bundles on curves with parabolic structure. *Math. Ann.*, 248:205–239, 1980.
[9] E. Meinrenken and C. Woodward. Hamiltonian loop group actions and Verlinde factorization. *Journal of Differential Geometry*, 50:417–470, 1999.
[10] Marcin Poźniak. Floer homology, Novikov rings and clean intersections. In *Northern California Symplectic Geometry Seminar*, volume 196 of *Amer. Math. Soc. Transl. Ser. 2*, pages 119–181. Amer. Math. Soc., Providence, RI, 1999.
[11] C. Teleman and C. Woodward. Parabolic bundles, products of conjugacy classes and Gromov-Witten invariants. *Ann. Inst. Fourier (Grenoble)*, 53(3):713–748, 2003.
[12] Katrin Wehrheim. *Uhlenbeck compactness.* EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2004.