Euler Characteristic
(and the Unity of Mathematics)

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SPLASH

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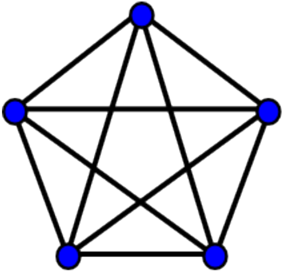
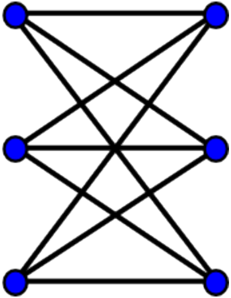
Graphs

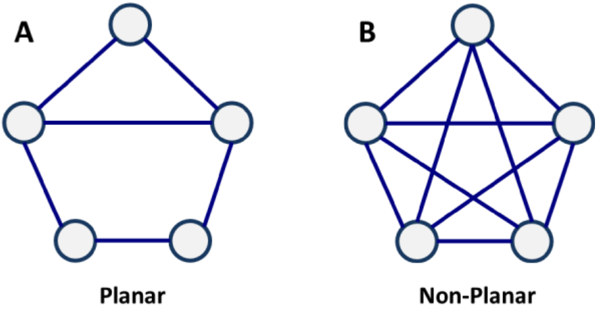
Collection of dots (vertices) connected by lines (edges)





Planar Graphs

Definition: A graph is *planar* if there are no edge crossings



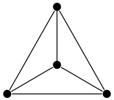
- Let V =number of vertices, E = number of edges, F = number of faces (including outside face)
- For Graph A: $V = 5, E = 6, F = 3$

Euler characteristic

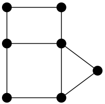
Definition: the Euler characteristic of a *planar* graph G is

$$\chi(G) = V - E + F$$

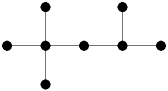
- Ex 1. $\chi(G) = 4 - 6 + 4 = 2.$



- Ex 2. $\chi(G) = 7 - 9 + 4 = 2.$



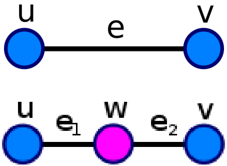
- Ex 3. $\chi(G) = 8 - 7 + 1 = 2.$



Euler's Theorem

Euler's theorem: All planar graphs have Euler characteristic 2!

- Ex. when subdivide an edge by adding a vertex,
  $V' - V = 1, E' - E = 1, F' - F = 0$ so $\chi(G') = \chi(G)$

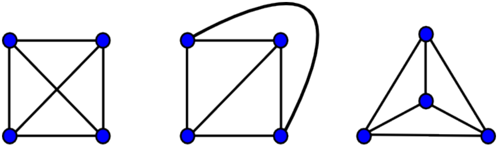


- Ex: when add an edge connected to two vertices,
  $V' - V = 0, E' - E = 1, F' - F = 1$ so $\chi(G') = \chi(G)$
- Ex: when add an edge connected to one vertex,
  $V' - V = 1, E' - E = 1, F' - F = 0$ so $\chi(G') = \chi(G)$
- So $\chi(G) = \chi(\text{single vertex}) = 1 + 1 = 2$

Non-planar graphs

Planarity is really an intrinsic property: a graph is planar if it *can be* draw so that no edge crossings.

Example:



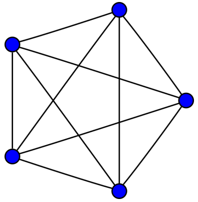
These all have the same underlying graph, with edges redrawn. Hence the graph is planar.

Non-planar graphs

Planar graphs have $e \leq 3v - 6$

- For any planar graph, $3f \leq 2e$
- Euler characteristic $2 = v - e + f$
- So $6 = 3v - 3e + 3f \leq 3v - 3e + 2e = 3v - e$
- Hence $e \leq 3v - 6$ for planar graphs

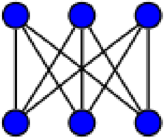
Ex 1. K_5 graph



$V = 5, E = 10$ so $10 \not\leq 3 \cdot 5 - 6$ so K_5 is not planar!

Non-Planar Graphs

Ex 2. $K_{3,3}$ graph

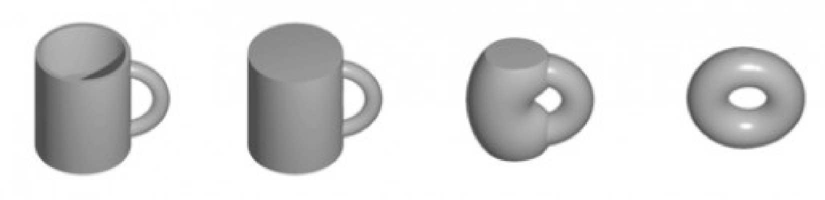


By similar argument, $K_{3,3}$ is non-planar.

Kuratowski's Theorem: a graph is planar if and only if it has no subgraphs that are (expansions of) $K_{3,3}$ or K_5

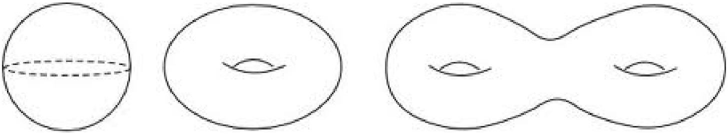
Topology

Topology: "the study of geometric properties and spatial relations unaffected by the continuous change of shape or size of figures."



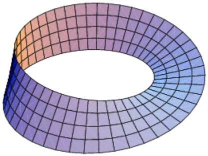
Surfaces

- All (orientable) surfaces without boundary are



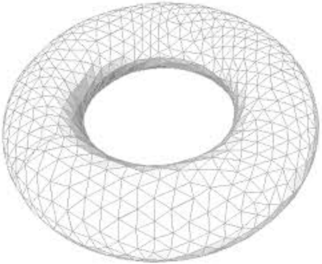
  i.e. spheres and multi-holed donuts.

- Example of non-orientable surface: Mobius strip



Euler characteristic of Surfaces

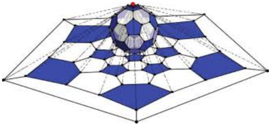
Triangulation of surfaces:



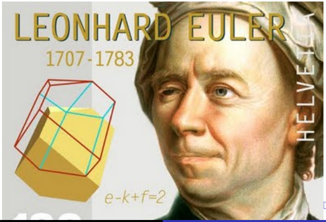
Definition: Euler $\chi(S, T)$ of a surface S is $V - E + F$ for some triangulation T of S

Euler characteristic of Surfaces

- Euler's Theorem implies that all spheres have Euler characteristic 2!

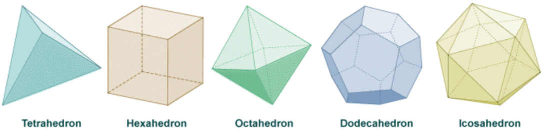


- Similar to proof of Euler's Theorem: $\chi(S)$ is independent of triangulation, i.e. Euler characteristic is a topological invariant!
- HW: what is Euler characteristic for donuts with g holes?
- HW: define Euler characteristic for *non-planar* graphs



Platonic Solids

Constructed from congruent regular polygons (equal side length and angles) with the same number of faces meeting at each vertex.



- Plato: each is associated to "classical elements" - Earth, air, water, and fire (fifth one?)
- Ex. Euler characteristic of tetrahedron: $4 - 6 + 4 = 2$
- Ex. Euler characteristic of octahedron: $6 - 12 + 8 = 2$

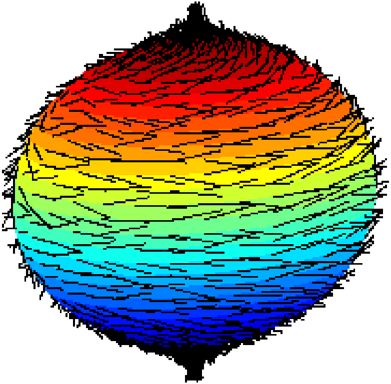
Classification of Platonic Solids

Theorem: These are the only Platonic solids!

- Suppose faces are n-gons and m edges meet at each vertex
- So $nF = 2E$ and $mV = 2E$
- Euler characteristic becomes $V - E + F = \frac{2E}{m} - E + \frac{2E}{n} = 2$ or $\frac{1}{m} + \frac{1}{n} = \frac{1}{E} + \frac{1}{2}$
- There are only five such pairs satisfying this equation $(m, n) = (3,3), (3,4), (3,5), (4,3), (5,3)$ and these all give Platonic solids (tetrahedron, hexahedron, dodecahedron, octahedron, icosahedron)

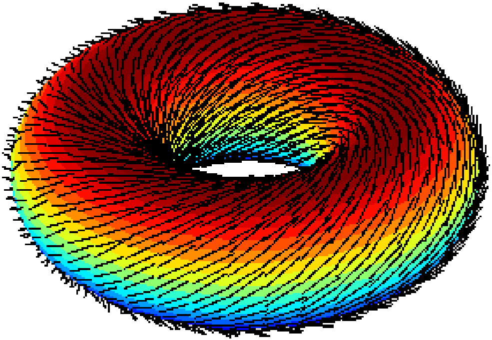
Hairy Ball Theorem

Theorem: Can't comb a hairy ball flat (two hairs will always be sticking out).



Or there is always a point on Earth where wind velocity is exactly zero (actually two points).

Hairy Donut Theorem?



Hairy Donuts can be combed.

"Combability" depends on topology.

Vector Fields

- Mathematicians use vector fields v
- At each point on a
- Ex. $I(S^2, v) = 2$
- Ex. $I(T^2, v) = 0$

Poincare-Hopf Theorem

Poincare-Hopf Theorem: for any vector field v

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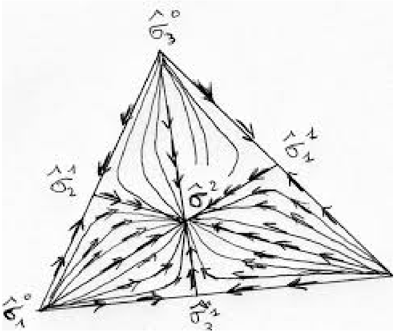
- Links topology and geometry; a priori vector fields has nothing to do with topology; if don't take alternating signs

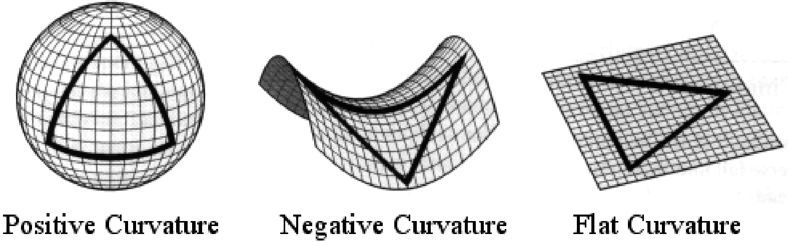
Construction of Vector Fields

Proposition: exists a vector field v on X with $I(X, v) = \chi(X)$



Geometry

Curvature



Not a topological invariant!

Gauss-Bonnet Theorem

X is a surface Gauss-Bonnet Theorem: $\int_X K(x) = \chi(X)$.

- Links geometry and topology!
- Right-hand-side seems like it depends on how put surface into space but actually independent!

Betti Numbers

- Euler characteristic are a shadow of more invariants
- To any space X, we can assign sequence of 'Betti numbers' $b_i(X) \geq 0$, where $i \geq 0$ is an integer; that are invariants
- Then $\chi(X) = \sum_{i \geq 0} (-1)^i b_i(X)$

Thank You!