Weinstein geometry of cotangent bundles

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Western Hemisphere Virtual Symplectic Seminar

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Handle anatomy

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  X^{2n} exact symplectic with contact boundary and isotropic sphere $\Lambda^{k-1} \subset \partial X^{2n} \implies$ new exact symplectic $X^{2n} \cup H^k_\Lambda$

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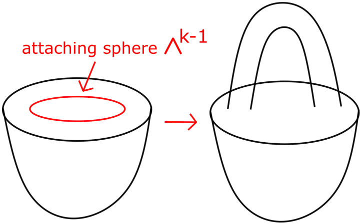

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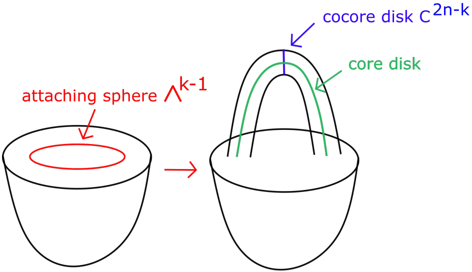
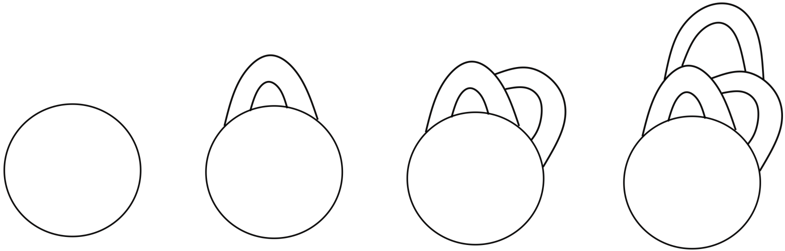


- Co-core of index n handle is Lagrangian disk with Legendrian boundary
- Index $k \leq n$ and handles of index less than n are "topological"

Weinstein domains

- **Weinstein domain** is result of iterated handle attachment to B^{2n}_{std} (0-handle), i.e. symplectic handlebody



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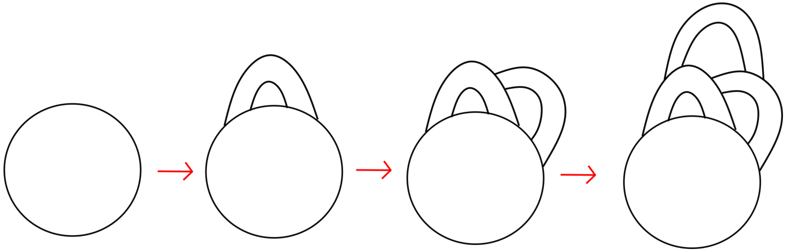

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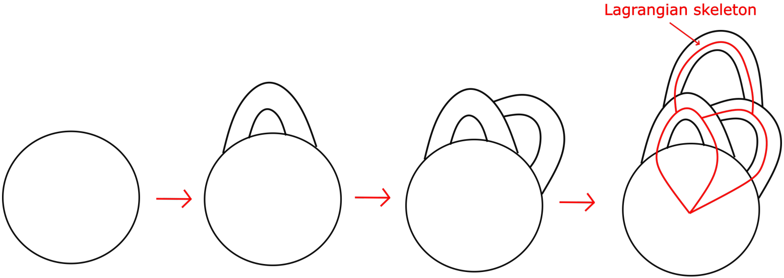

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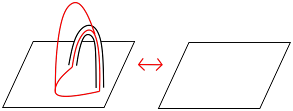
  2) Cancel/create handles 

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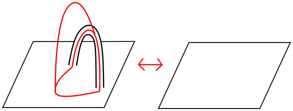
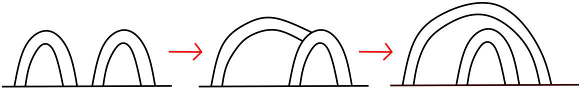
  2) Cancel/create handles 
  3) Handleslide 

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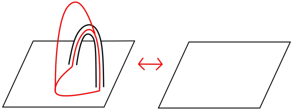
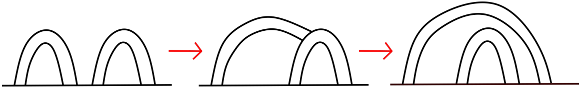
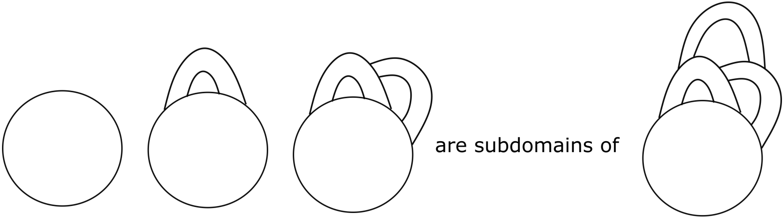
  2) Cancel/create handles 
  3) Handleslide 
- Same as smooth handle moves, except attaching sphere must be isotropic.
- **Question:** What do Weinstein presentations of X tell us about the symplectic geometry of X?

Weinstein subdomains

- Any Weinstein presentation has canonical collection of Weinstein subdomains - take union of handles up to some level



Question: What are all Weinstein subdomains, i.e. singular Lagrangians, of X?

Wrapped Fukaya category

- Wrapped Fukaya category $\mathcal{W}(X)$
  objects are (twisted complexes of) embedded exact Lagrangians $L \subset X$, closed or with Legendrian boundary $\partial L \subset \partial X$.
  morphisms are wrapped Floer cochains $CW^*(L, K)$

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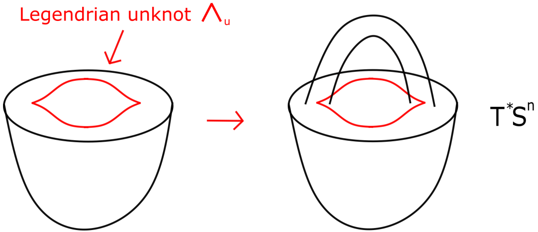
Theorem (Chantraine-Dimitroglou Rizell-Golovko-Ghiggini, Ganatra-Pardon-Shende)

If X^{2n} is Weinstein, the index n co-cores $C_1, \cdots, C_k$ generate $\mathcal{W}(X)$.

- **Generate:** any Lagrangian is isomorphic to a twisted complex (iterated cone) of co-cores, i.e. $\mathcal{W}(X) = Tw\ (C_1, \cdots, C_k)$
- **Question**: which twisted complexes are isomorphic to an embedded exact Lagrangian?

Cotangent bundles

- Focus on $T^*S^n_{std}$; most results hold for more general domains.
- Morse function on S^n with two critical points of index $0, n$ $\Longrightarrow$ Weinstein structure on T^*S^n with 2 handles of index $0, n$, i.e. $T^*S^n_{std} = B^{2n}_{std} \cup H^n_{\Lambda_u}$, Λ_u is Legendrian *unknot*.



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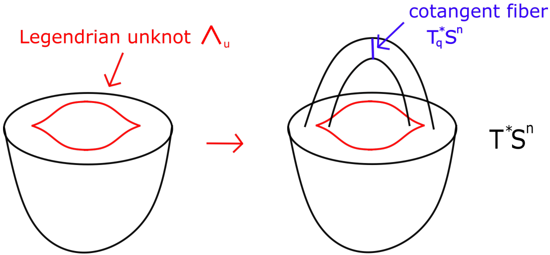
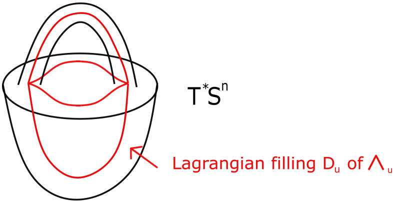


- One index n handle with co-core $T_q^*S^n$ $\implies \mathcal{W}(T^*S^n) \cong Tw\ T_q^*S^n \cong Tw\ C_{-*}(\Omega S^n)$.

Closed Lagrangians

- Λ_u has a Lagrangian disk filling $D^n_u \subset B^{2n}_{std}$ and zero-section $S^n \subset T^*S^n$ is union of D^n_u and core of $H^n_{\Lambda_0}$



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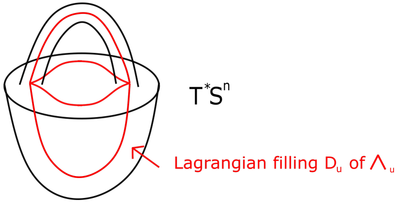

Theorem (Fukaya-Seidel-Smith, Nadler-Zaslow, Kragh, Abouzaid)

Any closed exact Lagrangian $L \subset T^*S^n$ is homotopy equivalent to S^n

$\implies$ any Lagrangian filling of Λ_u is homotopy equivalent to D^n.

- More generally, if $T^*S^n = B^{2n}_{std} \cup H^n_\Lambda$ is another presentation with *different* Λ, any filling of Λ is homotopy equivalent to D^n.

Exotic presentations

Theorem (L.)

If $n \geq 3$, exist infinitely many different Legendrian spheres $\Lambda_k^{n-1} \subset \partial B^{2n}_{std}$ so that $B^{2n}_{std} \cup H^n_{\Lambda_k}$ is Weinstein homotopic to $T^*S^n_{std}$. None are exact Lagrangian fillable.

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- False for $n = 2$: if $B^4_{std} \cup H^2_\Lambda = T^*S^2_{std}$, then Λ is the Legendrian unknot, i.e. all presentations are standard.
- Algebraic version: there are many different objects that generates $\mathcal{W}(T^*S^n)$ besides $T^*_q S^n$.
- Equivalently: Λ_k have different Chekanov-Eliashberg algebras but are derived Morita equivalent.

Exotic subdomains

Theorem (L. with Sylvan)

If $n \geq 5$, for any finite collection of primes P (possibly containing 0), there is a Weinstein subdomain $T^*S^n_P \subset T^*S^n$ so that

1) $\mathcal{W}(T^*S^n_P) \cong \mathcal{W}(T^*S^n)[\frac{1}{P}]$
2) $T^*S^n_P$ *is a Weinstein subdomain of* $T^*S^n_Q$ *if and only if* $Q \subset P$.
3) $T^*S^n_P$ *are all diffeomorphic to* T^*S^n

So $T^*S^n \supsetneq T^*S^n_2 \supsetneq T^*S^n_{2,3} \supsetneq T^*S^n_{2,3,5} \cdots \supsetneq T^*S^n_0 = T^*S^n_{flex}$

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  $\Longrightarrow T^*S^n_P$ has no (smooth) closed exact Lagrangians.
- **Abouzaid-Seidel:** there is an abstract Weinstein domain X_P so that $SH(X_P) \cong SH(T^*S^n)[\frac{1}{P}]$. Conjecturally $X_P = T^*S^n_P$. We show $T^*S^n_P$ are nested subdomains of T^*S^n.

Classifying subdomains

Theorem (L. with Sylvan)

Any Weinstein subdomain X of T^*S^n satisfies $\mathcal{W}(X) \cong \mathcal{W}(T^*S^n)[\frac{1}{P}]$ for some unique collection of primes P.

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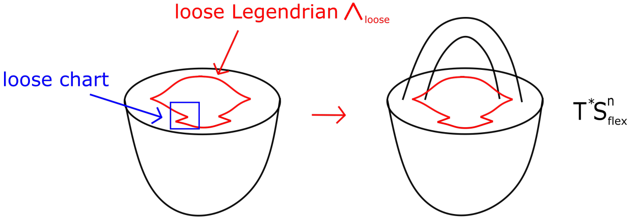
- Theorem holds only for cotangent bundles; even $T^*M \natural T^*N$ has subdomains T^*M and T^*N that are not prime localizations.
- In process, classify which twisted complexes in $\mathcal{W}(T^*S^n)$ are isomorphic to exact Lagrangian disks.

Flexible Weinstein domains

- **Flexible Weinstein domain:** all index n handles have Legendrian attaching spheres that are *loose*, i.e. zig-zag
- **Ex:** flexible cotangent bundle $T^*S^n_{flex}$, has $\mathcal{W}(T^*S^n_{flex}) \cong 0$



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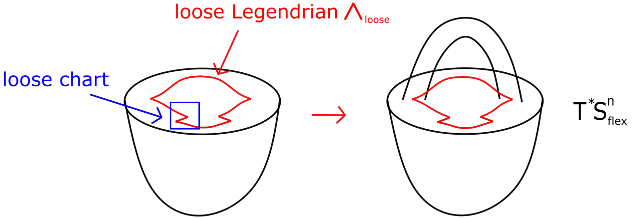

- **Ex:** *subcritical* domain with all handles of index *less than n*

Theorem (Cieliebak-Eliashberg, Murphy)

Let $n \geq 3$. If flexible Weinstein structures W_0, W_1 are homotopic through smooth handle moves (+framing), they are homotopic through Weinstein handle moves.

Flexible subdomains

- If $n \geq 3$ and X^{2n} is Weinstein with index n co-cores $C_1, \cdots, C_k$, then $X \backslash C_1 \coprod \cdots \coprod C_k$ has no index n handles, i.e. subcritical.

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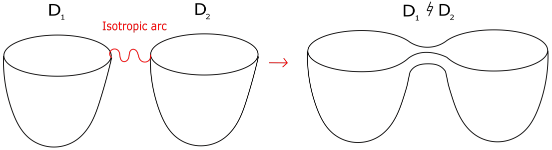
- **Boundary connected sum:** given two disjoint exact Lagrangians D_1, D_2 and isotropic arc from ∂D_1 to ∂D_2, can form boundary connected sum $D_1 \natural D_2$, new exact Lagrangian.



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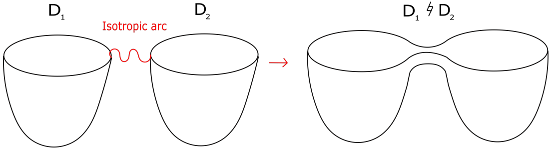

Theorem (L.)

If $n \geq 3$ and X^{2n} is Weinstein with index n co-cores $C_1, \cdots, C_k$, then $X \backslash C_1 \natural \cdots \natural C_k$ is a flexible domain.

- Equivalently: there is a flexible subdomain $V_{flex} \subset X^{2n}$ so that $X^{2n} = V_{flex} \cup H^n_\Lambda$ and co-core of H^n_Λ is $C_1 \natural \cdots \natural C_k$.

Weinstein presentations with few handles

- There is a flexible subdomain $V^{2n}_{flex} \subset X^{2n}$ so that $X^{2n} = V^{2n}_{flex} \cup H^n_\Lambda$.
- For V_{flex}, Weinstein handle moves are the same as smooth handle moves.

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Corollary (L.)

If $n \geq 3$, X^{2n} has a Weinstein presentation with at most two more Weinstein handles than the minimum number of smooth handles.

- Result is sharp: sometimes need two more Weinstein handles than smooth handles.
- Unknown if true in dimension $n = 2$.

Proof of Theorem

- **Theorem:** if X^{2n}, $n \geq 3$, has two co-cores C_1, C_2, then $X^{2n} \backslash C_1 \natural C_2$ is flexible.
- Need to realize $C_1 \natural C_2$ as co-core of some presentation.
- Idea: handleslides change co-cores by boundary connected sum.

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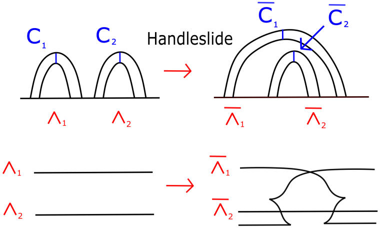

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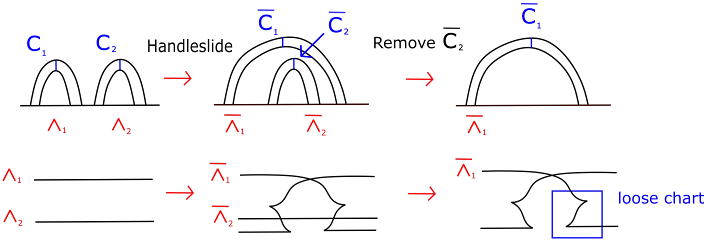
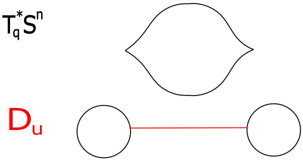


- **Proposition (L.):** $\overline{C_1} = C_1$ and $\overline{C_2} = C_1 \natural C_2$.
- So when remove $\overline{C_2} = C_1 \natural C_2$, have just $H^n_{\Lambda_1}$, which is loose.

Weinstein homotopy of T^*S^n



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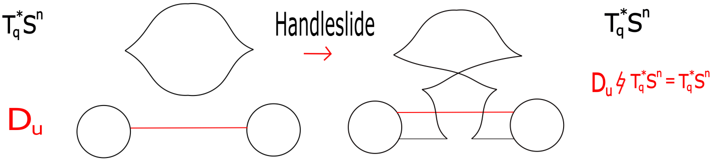

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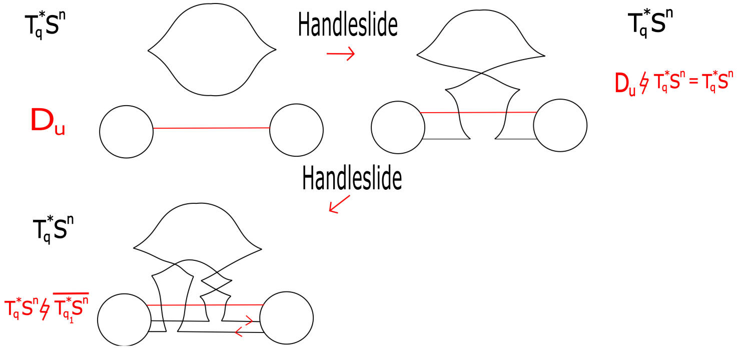

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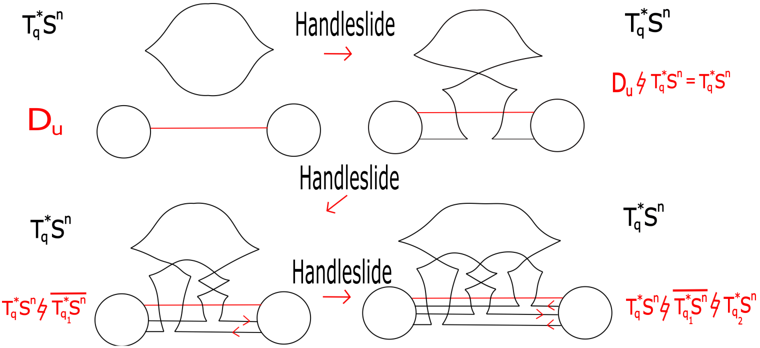

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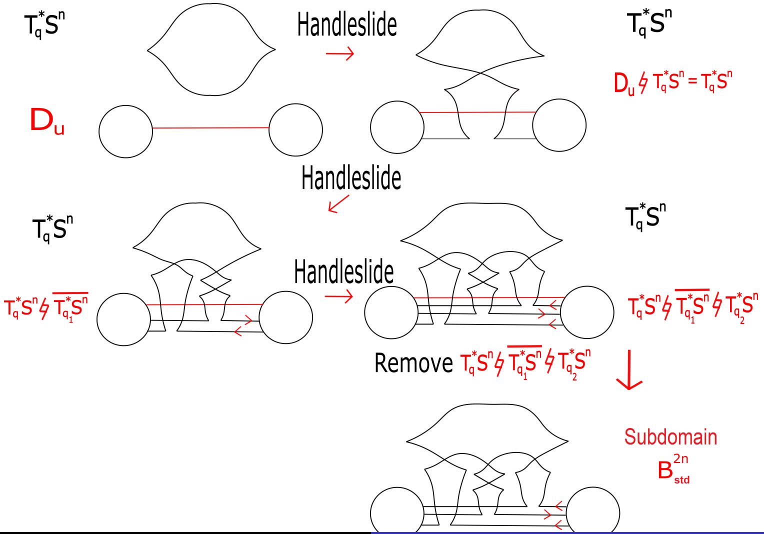

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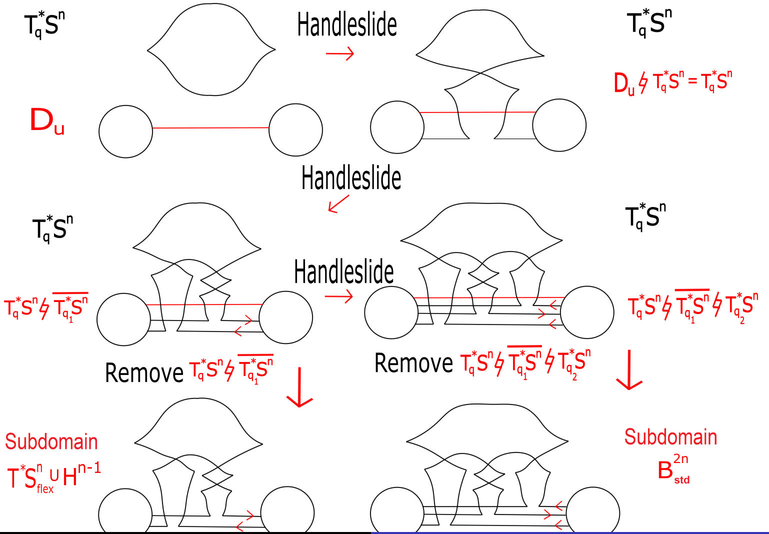

Exotic presentations for T^*S^n

- Exists $\Lambda_3 \subset \partial B^{2n}_{std}$ so that $B^{2n}_{std} \cup H^n_{\Lambda_3} = T^*S^n$ and co-core of $H^n_{\Lambda_3}$ is $C_{\Lambda_3} := T_q S^n \natural \overline{T^*_{q_1} S^n} \natural T^*_{q_2} S^n$
  $\implies \Lambda_3$ not isotopic to Λ_u and not exact fillable.

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- $T^*_q S^n \natural \overline{T^*_{q_1} S^n} \natural T^*_{q_2} S^n \cong T^*_q S^n \oplus T^*_{q_1} S^n[1] \oplus T^*_{q_2} S^n$ generates $\mathcal{W}(T^*S^n)$

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- $T^*_q S^n \natural \overline{T^*_{q_1} S^n} \natural T^*_{q_2} S^n \cong T^*_q S^n \oplus T^*_{q_1} S^n[1] \oplus T^*_{q_2} S^n$ generates $\mathcal{W}(T^*S^n)$
- For *any* category and object A, have $A \oplus A \oplus A[1]$ generates A:

$$Cone(A \oplus A \oplus A[1] \xrightarrow{\pi_{23}} A \oplus A \oplus A[1]) \cong A \oplus A[1]$$

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Algebraic flexibility

- **Geometric flexibility:** If $X^{2n}, n \geq 3$, has co-cores $C_1, \cdots, C_k$ and $C_1 \natural \cdots \natural C_k$ generates $H^n(X; \mathbb{Z})$, then X^{2n} has a presentation with a *single* co-core $C_1 \natural \cdots \natural C_k$

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- **Algebraic flexibility:** Let $\mathcal{C}$ be an arbitrary triangulated category.

Theorem (Thomason)

If $A_1, \cdots, A_k$ are generators of $\mathcal{C}$ and $A_1 \oplus \cdots \oplus A_k$ generate the Grothendieck group $K_0(\mathcal{C})$, then $A_1 \oplus \cdots \oplus A_k$ generate $\mathcal{C}$.

Grothendieck group

- How to link geometric and algebraic flexibility?

Theorem (L.)

If X^{2n} is Weinstein, there is a surjective homomorphism $H^n(X;\mathbb{Z}) \to K_0(\mathcal{W}(X))$ taking a n-cocycle to any Poincaré dual Lagrangian representative.

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- **Question:** How to construct subdomains of X^{2n}?
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- Hence to understand subdomains $X \subset T^*S^n$ suffices to understand Lagrangian disks as objects of $\mathcal{W}(T^*S^n)$.
- **Question:** What are Lagrangian disks $D^n \subset T^*S^n$?

Abouzaid-Seidel disks

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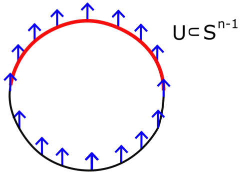
- For any smooth codimension 0 $U \subset S^{n-1}$, Abouzaid-Seidel constructed Lagrangian disk $D_U \subset T^*D^n \subset T^*S^n$.



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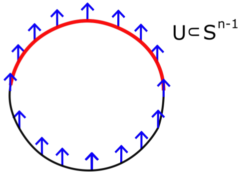
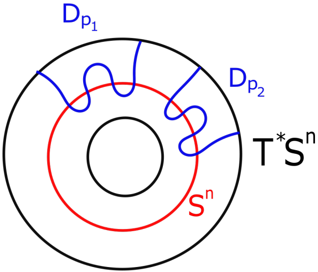


- $CW(S^n, D_U) \cong C^*(D^n, U) \cong \tilde{C}^{*-1}(U)$, reduced Morse cochains $\implies D_U \cong \tilde{C}^{*-1}(U) \otimes T_q^* S^n$ in $\mathcal{W}(T^*S^n)$.
- **Example:** if $n \geq 5$, exists Moore space $U_p \subset S^{n-1}$ so that $\tilde{C}^*(U) = \mathbb{Z}[-1] \xrightarrow{p} \mathbb{Z}[-2]$ $\implies D_U \cong T_q^* S^n[1] \xrightarrow{p} T_q^* S^n = Cone(p \cdot Id_{T_q^* S^n})$.

Constructing subdomains

- Set $T^*S^n_{p_1,\cdots,p_k} := T^*S^n \backslash (D_{p_1} \coprod \cdots \coprod D_{p_k})$ + flexible handles
- By construction, $T^*S^n_P \subset T^*S^n$ and $T^*S^n_P \subset T^*S^n_Q$ if $Q \subset P$



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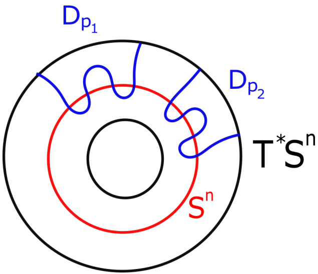

- $\mathcal{W}(T^*S_P) \cong \mathcal{W}(T^*S^n)/(D_{p_1}, \cdots, D_{p_k})$
- Killing $D_p \cong Cone(p \cdot Id_{T^*_q S^n})$ is same as making $p \cdot Id_{T^*_q S^n}$ an isomorphism, i.e. inverting p
  $\Longrightarrow \mathcal{W}(T^*S_P) \cong \mathcal{W}(T^*S^n)[\frac{1}{P}]$

Classifying subdomains

- To classify subdomains $X \subset T^*S^n$, suffices to classify isomorphism classes of disks $D \subset T^*S^n$ in $\mathcal{W}(T^*S^n)$.

Theorem (L. w/ Sylvan)

If $\pi_1(M) = H^1(L) = H^2(M) = 0$ and $i : L \hookrightarrow T^*M$ is a null-homotopic Lagrangian embedding, then L is isomorphic to $CW(M, L) \otimes T^*_q M$ in $\mathcal{W}(T^*M)$.

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- Every chain complex over $\mathbb{Z}$ splits into $\mathbb{Z}[1] \xrightarrow{k} \mathbb{Z}$ $\implies$ any null-homotopic L, e.g. disk D, is isomorphic to $\oplus_i D_{k_i}$ $\implies$ any Weinstein subdomain $X \subset T^*S^n$ has $\mathcal{W}(X) \cong \mathcal{W}(T^*S^n)/D \cong \mathcal{W}(T^*S^n)/(\oplus_i D_{k_i}) \cong \mathcal{W}(T^*S^n)[\frac{1}{P}]$, where P is set of primes dividing k_i.

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