Homework 3 Solution

Exercise 1.3.5. Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$, together with the segments of the vertical lines $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ inside the square. Show that for every covering space $\widetilde{X} \to X$, there is some neighborhood of the left edge of X that lifts homeomorphically to $\widetilde{X}$. Deduce that X has no simply-connected covering space.

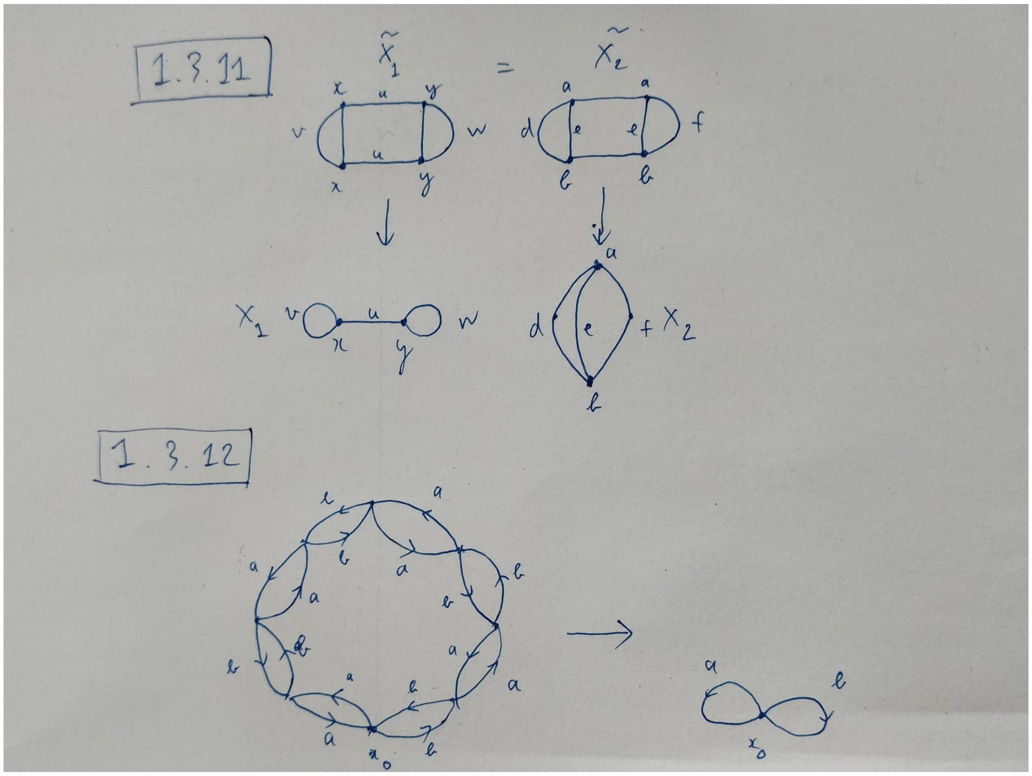
Proof. Let $L \times \{0\}$ be the left edge. For each point $x \in L \times \{0\}$, there is an open neighborhood U_x of X around x whose lifts $p^{-1}(x)$ are disjoint copies of U_x. Since L is compact, it has a finite subcover $U_{x_1}, \ldots, U_{x_n}$ for $x_1, \ldots, x_n \in L$. We will show that $U_{x_1} \cup \cdots \cup U_{x_n}$ is lifted to disjoint copies of itself in $\widetilde{X}$.

Since L is connected, two of $U_{x_1}, \ldots, U_{x_n}$ must intersect. Assume $U_{x_1} \cap U_{x_2} \neq \emptyset$, and pick $y \in U_{x_1} \cap U_{x_2}$. Pick a neighborhood $U' \subset U_{x_1} \cap U_{x_2}$ around y that is evenly covered by p; each of its disjoint copies in $p^{-1}(U')$ will be the intersection of a copy of U_{x_1} and a copy of U_{x_2}. Hence, $p^{-1}(U_{x_1} \cup U_{x_2})$ is a disjoint union of homeomorphic copies of $U_{x_1} \cup U_{x_2}$. If $n > 2$, then $U_{x_1} \cup U_{x_2}$ must intersect some U_{x_i} non-trivially, and the same argument shows that $U_{x_1} \cup U_{x_2} \cup U_{x_i}$ is evenly covered by p. We can keep doing this until $U_{x_1} \cup \cdots \cup U_{x_n}$ is evenly covered by p.

Using the Tube Lemma, there exists $\epsilon > 0$ such that X restricted to the interval $[0, \epsilon)$ of the x-axis is contained in $U_{x_1} \cup \cdots \cup U_{x_n}$. This subspace contains infinitely many loops that are not homotopic to the identity; for example, loops that are the sides of the rectangle $L \times [0, 1/n]$ with $n > 1/\epsilon$. However, we showed that this subspace is lifted homeomorphically to $\widetilde{X}$, hence $\widetilde{X}$ also contains non-trivial loops. In other words, X has no simply-connected covering space. $\square$

Exercise 1.3.11. Construct finite graphs X_1, X_2 having a common finite-sheeted covering space $\widetilde{X_1} = \widetilde{X_2}$, but such that there is no space having both X_1 and X_2 as covering spaces.

Proof. See picture for the spaces. It is clear that the maps described in the picture are covering spaces. If there is a graph Y covered by both X_1 and X_2, then it either has 1 or 2 vertices. If Y has 2 vertices, then this is a one-sheeted cover, which forces Y to be either X_1 or X_2. However, neither of these spaces cover each other (if X_2 cover X_1, for example, then x would be covered by a or b, but in this case the edge v cannot be lifted). If Y has 1 vertex, then that vertex has degree 3, the same as that of all vertices of X_1, X_2, but a graph cannot have total degree being an odd number, contradiction. $\square$



Exercise 1.3.12. Let a, b be the generators of $\pi_1(S^1 \vee S^1)$. Draw a picture of the covering space of $X = S^1 \vee S^1$ corresponding to the normal subgroup N generated by a^2, b^2 and $(ab)^4$, and prove that this covering space is the correct one.

Proof. See the picture for the covering space $p : \widetilde{X} \to X$. Note that $a^2, b^2, (ab)^4 \in p_*(\pi_1(\widetilde{X}, x_0))$, since we have loops corresponding to those elements in $\widetilde{X}$. Moreover, this covering space is normal since we have deck transformations that are either rotation by multiple of $\pi/4$, or rotation by $\pi/8$ plus reflection. Thus, $p_*(\pi_1(\widetilde{X}, x_0))$ is normal in $\pi_1(X, x_0)$, and so $N \subseteq p_*(\pi_1(\widetilde{X}, x_0))$. The other inclusion follows by noting that all loops in $\widetilde{X}$ are trivial in $\pi_1(X, x_0)/N$, e.g: $abba = aa = 1$, $abaaba = abba = 1$, etc. $\square$