Inverting Primes in Weinstein geometry

Joint w Sylvan, Tanaka

I Localization in topology

Sullivan, Quillen... for any set of primes P,
there is a functor

$$( )_P : \text{Spaces} \to \text{Spaces}$$

"CW/htpy, $\pi_1 = 0$"

1) invariants localize $H_i(X_P;\mathbb{Z}) \cong H_i(X;\mathbb{Z}) \otimes \mathbb{Z}[\tfrac{1}{P}]$

2) construction of X_P depends on CW presentation of X - but homotopy type independent

3) idempotent $(X_P)_P \cong X_P$

Ex. $S^1_P =$  $\dots$

p times winding

mapping telescope

II Symplectic localization

· symplectic analog of CW complex is Weinstein domain X^{2n}

cocores are Lagrangian

isotropic attaching spheres

B^{2n}

} critical handles index n

} subcritical handles index $< n$ essentially topological

· P set of primes

Thm (Abouzaid–Seidel) given X^{2n}, $n \geq 6$, exists $X_P'^{2n}$ diffeo to X s.t.

$$SH(X_P) \cong SH(X) \otimes \mathbb{Z}[\tfrac{1}{P}]$$

symplectic cohomology

$SH(X)$ not iso $H^\cdot(X)$

$H^\cdot(X)$ is $H^\cdot(X_P)$

Idea: modify vanishing cycles in Lefschetz fibration for X

- not clear if X_p is independent of Lef fib / Weinstein presentation of X
- not clear if $(X_p)_p \cong X_p$

<u>Thm</u> (Cieliebak-Eliashberg, Murphy)
given X^{2n}, $n \geq 3$, exists X^{2n}_{flex} Flexible
- $W(X_{flex}) \cong 0$, wrapped Fukaya
- X_{flex} is diffeo to X
- h-principle: if X, Y diffeomorphic, X_{flex}, Y_{flex} are symplectomorphic

$\Rightarrow$ X_{flex} is independent of pres of X and $(X_{flex})_{flex} \simeq X_{flex}$ symplectomorphism or Weinstein homotopy

- $(\ )_{flex}$ is local modification of attaching spheres

<u>Ex</u>, $T^*S^n \longrightarrow T^*S^n_{flex}$

$\int$ Weinstein homotopy

Weinstein htpy by Murphy's

think singular Lagrangian

Thm (L. with Sylvan) given X^{2n}, $n \geq 5$, exists

- a Weinstein subdomain $X^{2n}_P \subset X^{2n}$ diffeo to X
- $X_P \subset X_Q$ if $P \supset Q$ or $0 \in P$
- wrapped Fukaya $Tw\mathcal{W}(X_P) \simeq Tw\mathcal{W}(X) \otimes \mathbb{Z}[\frac{1}{P}]$
- local modification of attaching spheres
- $X_0 = X_{flex}$, $\mathcal{W}(X)[\frac{1}{0}] \simeq 0 \simeq \mathcal{W}(X_{flex})$

Ex.

$$
\begin{array}{ccccc}
 & & T^*S^n & & \\
 & \cup & \cup & \cup & \\
 & T^*S^n_2 & T^*S^n_3 & T^*S^n_5 & \cdots \\
 & \cup & & \cup & \\
 & T^*S^n_{2,3} & & T^*S^n_{3,5} & \cdots \\
 & & \cup & & \\
 & & T^*S^n_{2,3,5} & \cdots & \\
 & & \cup & & \\
 & & \vdots & & \\
 & & T^*S^n_{flex} & &
\end{array}
$$

fiber square

$$
\begin{tikzcd}
\mathcal{W}(T^*S^n) \arrow[r] \arrow[d] & \mathcal{W}(T^*S^n)[\frac{1}{3}] \arrow[d] \\
\mathcal{W}(T^*S^n)[\frac{1}{2}] \arrow[r] & \mathcal{W}(T^*S^n)[\frac{1}{2,3}]
\end{tikzcd}
$$

- order cannot be reversed

if $T^*S^n_2 \subset T^*S^n_{2,3}$, then Viterbo ring map

$$SH(T^*S^n_{2,3}; \mathbb{Z}/3) \not\to SH(T^*S^n_2; \mathbb{Z}/3)$$

$$\begin{matrix} \parallel & & \wr\!\parallel \\ 0 & & SH(T^*S^n; \mathbb{Z}/3) \\ & & \neq \\ & & 0 \end{matrix}$$

<u>Thm</u> (L. w Sylvan) if M^n is simply-conected and spin, then any Weinstein subdomain $X \subset T^*M$ has $\mathrm{Tw}\,W(X) \cong \mathrm{Tw}\,W(T^*M_P)$ for some set of primes P

- not true in general that any $X_0 \subset X$ has $W(X_0) \cong W(X)[\frac{1}{p}]$

<u>Ex,</u> $T^*N \subset T^*N \natural T^*M$

<u>Q,</u> how unique is X_P?

don't affect Fukaya cat

Two "symplectically trivial" operations

- subcritical handle attachment, h-principle

$$X^{2n} \rightsquigarrow X^{2n} \cup H^k, \quad k < n$$

- stabilization sector

$$X^{2n} \longrightarrow X^{2n} \times T^* D^1$$

<u>Thm</u> (L. w Sylvan, Tanaka)

- if X, X' are symplectomorphic then X_p, X'_p are symplectomorphic up to stabilization and subcritical handles

- $(X_p)_p \cong X_p$ up to stabilization and subcritical handles

- functoriality → construct $\mathrm{Wein}^{stab}[\mathrm{sub}^{-1}]$

III Construction

- given a Lagrangian disk $D^n \subset X^{2n}$, can form subdomain $X \setminus D \subset X$

carving out

twisted complexes

Thm (Ganatra–Pardon–Shende)

$\mathrm{Tw}\, W(X \setminus D) \simeq \mathrm{Tw}\, W(X)/D$

categorical localization by object D

and Lagrangian cocores $C_x \subset X^n$ generate, i.e. $\mathrm{Tw}\, W(X) \simeq \mathrm{Tw}\, C_x$.

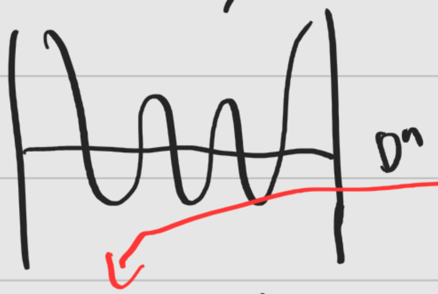
Abouzaid–Seidel disks

- given smooth subdomain $U \subset S^{n-1}$, $D^n_U \subset T^*D^n$ such that

reduced sing cochains, twisted complex

$D_U \simeq \tilde{C}^*(U) \otimes T_0 D^n$ in $\mathrm{Tw}\, W(T^*D^n)$

• $D_u = \Gamma_{df_u}$, $f_u : D^n \to \mathbb{R}$, $\lim_{x \to u \subset S^{n-1}} f(x) = -\infty$



• take $u \subset S^{n-1}$ to be p-Moore space

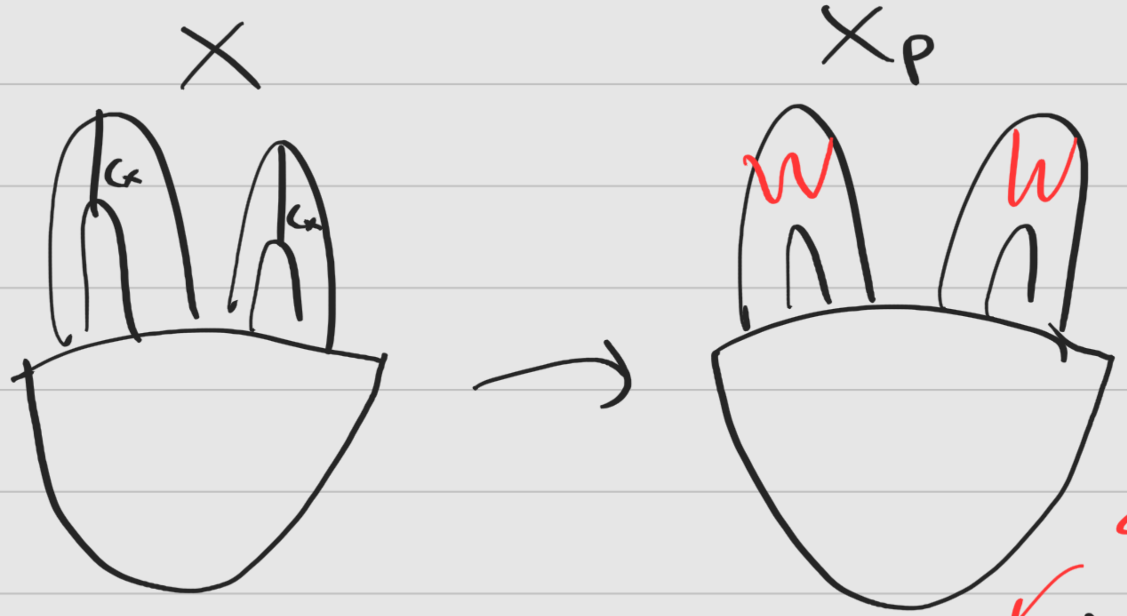
so $\tilde{C}^*(u) \cong \mathbb{Z}[1] \xrightarrow{p} \mathbb{Z}$, call D_p

u :  $S^1 \cup_p D^2 \hookrightarrow \mathbb{R}^5 \hookrightarrow S^5$

$\hookrightarrow S^4$

Carving out

• remove D_p near cocores C_x of X



(and reattach flexible handles) so that X_p, X are diffeo

• then $W(X_p) = W(X \setminus D_p)$

$\overset{GPS}{\cong} W(X)/D_p$

$\cong W(X)/\{\mathbb{Z}[1] \xrightarrow{p} \mathbb{Z}\} \otimes C_x$

$$\overset{\text{co... generate}}{\simeq} W(X) / \{\mathbb{Z}[1] \xrightarrow{p} \mathbb{Z}\} \otimes W(X)$$

$$\simeq W(X)[\tfrac{1}{p}]$$

forcing $\{\mathbb{Z}[1] \xrightarrow{p} \mathbb{Z}\}$ to be acyclic

$\leftrightarrow$ making p be invertible

IV Classifying subdomains

$X \subset T^*M$ Weinstein subdomain

$\leftrightarrow$ exists disks D_i s.t.

$T^*M \setminus D_i = X \cup$ subcritical handles

$\rightsquigarrow \mathrm{Tw}\, W(X) \overset{\mathrm{Tw}}{\simeq} W(T^*M) / D_i$ for some Lagrangian disks D_i

$\mathrm{Tw}\, T_q^\infty M \simeq \mathrm{Tw}\, C_{-*}(\Omega M)$ is

$$W^{\text{disks}}(T^*M) \hookrightarrow \overset{\text{embedded}}{W(T^*M)} \hookrightarrow \mathrm{Tw}\, W(T^*M)$$

Q. which twisted complexes are iso to actual Lagrangians in X?

Ex. $S^n \simeq T_q^* S^n[n] \xrightarrow{\gamma} T_q^* S^n$, $\gamma \in HW^{1-n}(T_q^* S^n, T_q^* S^n)$ ← higher degree

$D_2 \cong T_2^* S^n[1] \xrightarrow{2} T_2^* S^n \qquad 2 \in HW^0(T_2^* S^n, T_2^* S^n)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; 2 \cdot \mathrm{Id}$

$\mathbb{Z}$- chain complex $\otimes T_2^* S^n$

Thm (L. w Sylvan) If M is spin, simply-connected, any Lagrangian embedded, exact $L \subset T^*M$ that is null-homotopic is isomorphic to $C^* \otimes T_2^* M$

topological

$\mathbb{Z}$-chain complex

Ex image of $W(T^*S^n) \hookrightarrow Tw\, W(T^*S^n)$

$n \geq 5$, $\quad T_2^* S^n[n] \xrightarrow{\gamma} T_2^* S^n \cong S^n$

$\qquad C^* \otimes T_2^* S^n \cong \natural D_u^n$

· not all twisted complexes geometric

Ex. $\quad S^n[1] \xrightarrow{5} S^n$

Post-talk notes

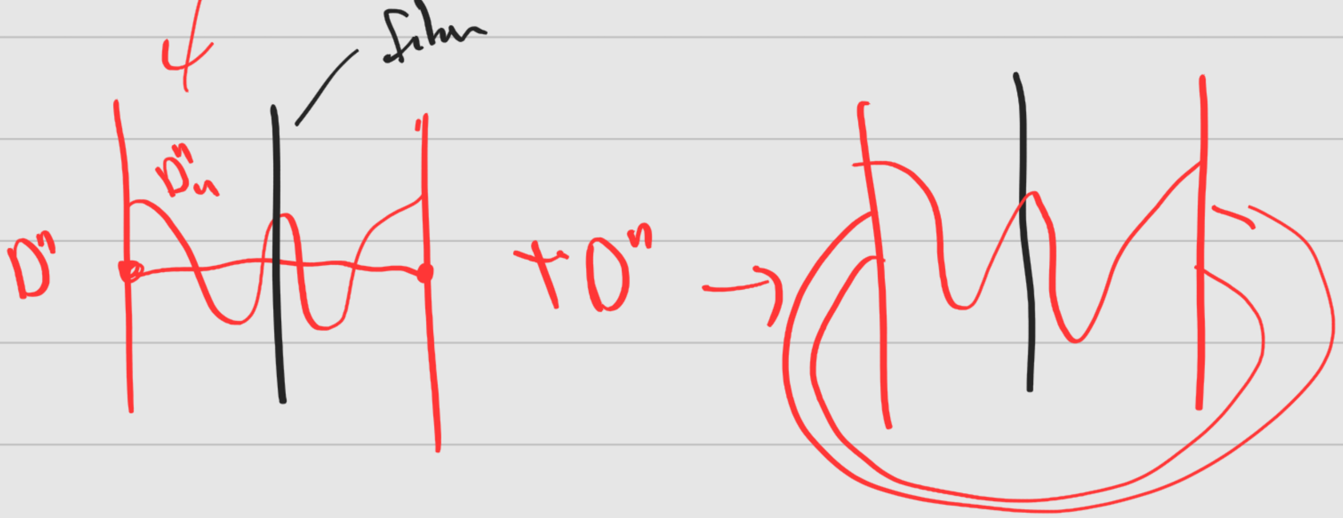
$H_\cdot(S^n, \{f \ll 0\})$

$\cong$

$WH(S^n, \Gamma_{df}) \overset{\text{Floer}}{\cong} H^*_{Morse}(f)$

$WH(S^n, T_0^* S^n) \cong \mathbb{Z}$

$WH(S^{n}; D_u) \cong H^{*}(U)$



$\partial(T^*D^n) \supset \partial D^n$ = Legendrian unknot

∂D^n_u = Legendrian unknot

$\partial D^n, \partial D^n_u$ are Legendrian linked

U = neighborhood of Morse space in S^{n-1}

$$TD^n \hookrightarrow TS^n$$
$$T^*_0 D^n \longrightarrow T^*_q S^n$$
$$D^n \hookrightarrow S^n$$

remove this.