Lecture 6 02/06/20

Last time:

- covering spaces $\begin{array}{c}\tilde{X}\\ \downarrow p\\ X\end{array}$
- homotopy lifting property
- $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective

Today classifying covering spaces

Def. a map between covering spaces $(\tilde{X}_1, p_1), (\tilde{X}_2, p_2)$ of X is a map $f : \tilde{X}_1 \to \tilde{X}_2$ so that

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

commutes

$$\Rightarrow \quad f : p_1^{-1}(x) \to p_2^{-1}(x)$$

- if f is a homeomorphism, f is called an isomorphism

Q1 given X, what are all covering spaces of X? (up to isomorphism)

Q2 given Y, what are all X so that Y covers X?

• focus first on Q1

Prop.1 if $f:(\tilde{X}_1,\tilde{x}_1)\xrightarrow{\sim}(\tilde{X}_2,\tilde{x}_2)$ covering space isomorphism

$$\begin{tikzcd}(\tilde{X}_1,\tilde{x}_1) \arrow[rr, "\sim"] \arrow[dr, "p_1"'] & & (\tilde{X}_2,\tilde{x}_2) \arrow[dl, "p_2"] \\ & (X,\tilde{x}) & \end{tikzcd}$$

then $(p_1)_*\pi_1(\tilde{X}_1)=(p_2)_*\pi_1(\tilde{X}_2)\subset\pi_1(X)$

Pf. $(p_1)_*\pi_1(\tilde{X}_1)=(p_2\circ f)_*\pi_1(\tilde{X}_1)=(p_2)_*f_*\pi_1(\tilde{X}_1)$

$=(p_2)_*\pi_1(\tilde{X}_2)$ since $f_*:\pi_1(\tilde{X}_1)\xrightarrow{\sim}\pi_1(\tilde{X}_2)$ $\square$

Thm there is a 1-to-1 correspondence between based covering spaces of X and subgroups of $\pi_1(X,x_0)$

$(\tilde{X},\tilde{x}_0)\xrightarrow{p}(X,x_0)\longrightarrow p_*\pi_1(\tilde{X},\tilde{x}_0)\subset\pi_1(X,x_0)$

$(X_G,\tilde{x}_0)\longrightarrow(X,x_0)\longleftarrow G\subset\pi_1(X,x_0)$

Rmks

1) there is a unique covering space $(\tilde{X},\tilde{x}_0)\xrightarrow{p}(X,x_0)$ with $\pi_1(\tilde{X})=0$, called universal cover corresponding to subgroup $0\subset\pi_1(X)$

2) $H\subset G\iff X_H$ covers X_G

Pf. $\Rightarrow$ exists a cover $(X_G)_H\xrightarrow{p}X_G$ with π_1 image $H\subset G$ and so, $(X_G)_H\to X_G\to X$ has image $H\subset\pi_1(X)$, so $(X_G)_H\cong X_H$

$\Rightarrow$ universal cover covers all covers of X

3) if change basepoint $\tilde{x}_0\in p^{-1}(x_0)$ to $\tilde{x}_0'$ in $\tilde{X}$ then $p_*\pi_1(\tilde{X},\tilde{x}_0')=g\,p_*\pi_1(\tilde{X},\tilde{x}_0)\,g^{-1}$, $g\in\pi_1(X)$

$\Rightarrow$ there is a 1-to-1 correspondence

covering spaces of X $\longleftrightarrow$ <u>conjugacy</u> classes of subgroups of $\pi_1(X, x_0)$

4) X should be path-connected, locally-path-connected, semi-localy-simply-connected, for any $x \in X$ exists $U \ni x$ so that $\pi_1(U) \to \pi(X)$ is 0 map

<u>Uniqueness</u>

<u>Prop.</u> (lifting criterion) given $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ <u>covering</u> and $(Y, y_0) \xrightarrow{f} (X, x_0)$, exists <u>unique</u> lift

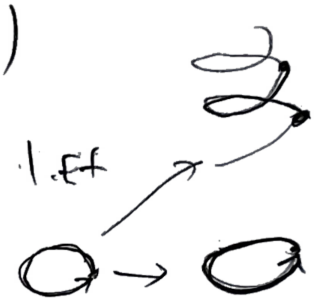
$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd}$$

iff $f_* \pi_1(Y) \subset p_* \pi_1(\tilde{X})$

<u>Ex.</u>

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ S^1 \arrow[r, "Id"] & S^1 \end{tikzcd}$$

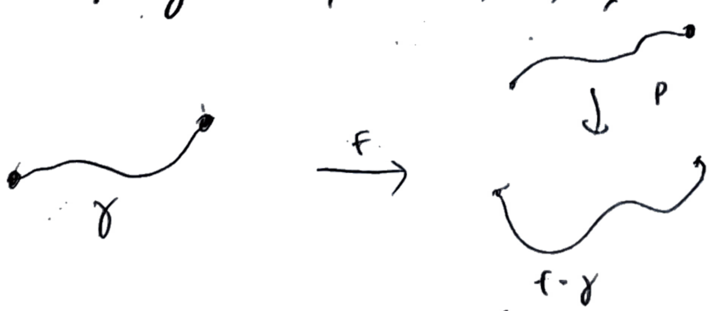
has no lift



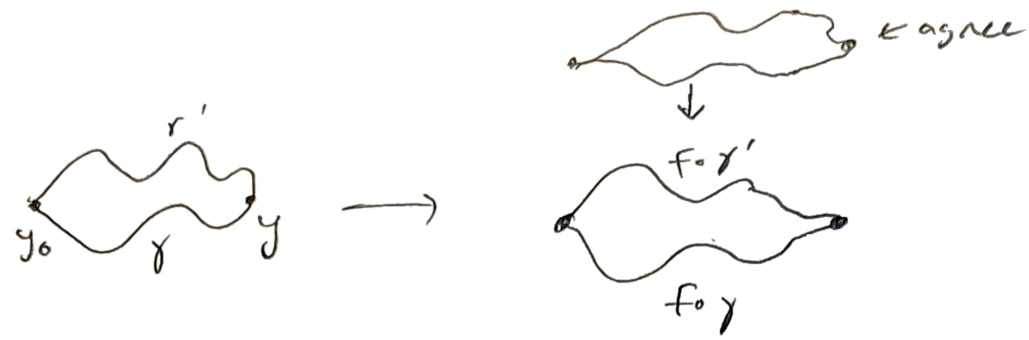
<u>Pf.</u> . use path-lifting property

. given any $y \in Y$, take path $\gamma : [0,1] \to Y$
$f(0) = y_0$, $f(1) = y$

and take lift of $f \circ \gamma : [0,1] \to X$ to $\widetilde{f \circ \gamma} : [0,1] \to \tilde{X}$, define $\tilde{f}(y) = \widetilde{f\circ\gamma}(1)$



- need to show well-defined-ness



$$\gamma(\gamma')^{-1} \in \pi_1(Y) \longrightarrow (f\circ\gamma)(f\circ\gamma')^{-1} \in f_*\pi_1(Y) = p_*\pi_1(\widetilde{X})$$

$$\Longrightarrow \widetilde{f\circ\gamma} \circ \widetilde{f\circ(\gamma')^{-1}} \text{ loop in } \widetilde{X} \Longrightarrow \widetilde{f\circ\gamma}(1) = \widetilde{f\circ\gamma'}(1)$$

<u>Rmk</u> false if $\widetilde{X} \to X$ not a covering space

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ S^2 \arrow[ur, dashed] \arrow[r, "Id"] & S^2 \end{tikzcd}$$

lift does not exist

<u>Uniqueness of lift:</u>

$$\begin{tikzcd} & \widetilde{X} \arrow[d] \\ Y \arrow[ur, "\widetilde{f_1}", shift left] \arrow[ur, "\widetilde{f_2}"', shift right] \arrow[r, "f"'] & X \end{tikzcd}$$

- $\widetilde{f_1}(y_0) = \widetilde{f_2}(y_0) = \widetilde{x_0}$,
- let $U \subset X$ containing y_0 so that $p^{-1}(U) = \sqcup U_\alpha$
- then exists $V \subset Y$ so that $f(V) \subset U$

and so $\widetilde{f_1}(V) \subset U_{\alpha_1}$, $\widetilde{f_2}(V) \subset U_{\alpha_2}$

$\Longrightarrow \alpha_1 = \alpha_2 \Longrightarrow \widetilde{f_1} = \widetilde{f_2}$ on V

and can cover Y by such V □

<u>Pf of injectivity of correspondence</u>

$\widetilde{X}_1 \xrightarrow{p_1} X$, $\widetilde{X}_2 \xrightarrow{p_2} X$ and $(p_1)_*\pi_1(\widetilde{X}_1) = (p_2)_*\pi_1(\widetilde{X}_2)$

- so exists lifts

$$\begin{tikzcd} & \widetilde{X}_2 \arrow[d, "p_2"] \\ \widetilde{X}_1 \arrow[ur, dashed, "f"] \arrow[r, "p_1"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \widetilde{X}_1 \arrow[d] \\ \widetilde{X}_2 \arrow[ur, dashed, "g"] \arrow[r] & X \end{tikzcd}$$

and by uniqueness $f\circ g = Id$, $g\circ f = Id$ ■