Mapping Class Groups and Lefschetz Fibrations

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Abstract

The connections between symplectic 4-manifolds, Lefschetz fibrations, and mapping class groups allow the use of symplectic techniques to prove statements about mapping class groups of surfaces, by constraining the set of allowable monodromies. In this paper we set out to prove these statements more directly. For planar surfaces, we exhibit an elementary proof that any factorization of a boundary monodromy obtained by a sequence of positive stabilizations of the identity into positive Dehn twists must have the same number of factors. Further, motivated by Smith's use of hyperbolic methods to show that there are no factorizations of the identity for surfaces with boundary [Sm], we demonstrate that a negative stabilization can never be factorized as a product of positive Dehn twists. [1]

Contents

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1 Introduction

In this paper, we are interested in two mathematical objects; the first are the mapping class groups of surfaces, which are groups of isotopy classes of homeomorphisms on 2-manifolds; the second are Lefschetz fibrations; a Lefschetz fibration is a map from a 4-manifold to a 2-manifold, which is a fibration (almost everywhere) except at finitely many points. These two objects are closely related: we can associate elements of the mapping class group of the fiber to monodromies on the corresponding Lefschetz fibration, and we can construct a Lefschetz fibration from the mapping class group factorization of a given surface.

Lefschetz fibrations also have a separate connection to symplectic 4-manifolds. Lefschetz fibrations can be thought of as a topological description of symplectic 4-manifolds, because, roughly speaking, symplectic 4-manifolds are manifolds that admit a Lefschetz fibration. This connection means that results concerning Lefschetz fibrations have been proven using symplectic geometry methods (see [Gom], [Don],[Ent], [El], [Pr] and [OS]). We are interested in two particular results, which are stated precisely in theorem 3.1 and theorem 3.5 in section 3. Theorem 3.1 states that if a 4-manifold with a Lefschetz structure has the same boundary as that of $\bar{\Sigma} \times D^2$ ($\bar{\sigma}$ is a surface with boundary), then the manifold must be diffeomorphic to $\bar{\Sigma} \times D^2$. Theorem 3.5 states that 4-manifolds can be classified as tight or overtwisted, and 4-manifold with a Lefschetz structure (which means that they have a symplectic filling) are tight (the two results will be explained with more detail later in the paper).

Because of the relation between Lefschetz fibrations and mapping class groups, the two theorems mentioned above tell us properties of mapping class group factorizations. In particular, theorem 3.1 tells us two properties of a mapping class group of surface with boundary (theorems 3.2 and theorem 3.4): identity cannot be factorized into positive Dehn twists, and a monodromy arising from stabilization can only be factorized into positive Dehn twists. Theorem 3.5 tells us that, for a surface with boundary, a monodromy arising from a negative stabilization cannot be factorized into positive Dehn twists (theorem 3.6). However, because the results are proven with symplectic geometry, their proofs are not very explicit in the context of mapping class groups.

Thus, to find elementary explanations of these consequences of symplectic geometry, we attempted to prove the consequences using only methods from mapping class groups. Motivated by [PV], we studied the mapping class group of D_n, the disk with n holes, as an example, and proved the consequences for D_n. We were then motivated to study abelianizations of mapping class groups, which gave some limited insight. Motivated by [Sm] and [Hon], we also studied the hyperbolic metric on the universal cover of a surface, and we managed to obtain some results there. Our results will be presented later.

In this paper, we will first provide background information on the mapping class group and Lefschetz fibrations, with the aim to make the paper self-contained. Then we will state the results that come from symplectic geometry, and explain how they give consequences in the mapping class group. Finally, we will present our investigation of these consequences using elementary methods.

2 Background

In this section, we will provide background on the mapping class group and Lefschetz fibrations. We will explain how to identify mapping class group factorizations from Lefschetz fibrations, and how to construct Lefschetz fibrations from mapping class group factorizations. Then we will explain the Hurwitz problem, an open problem concerning the classification of Lefschetz fibrations (for more on the Hurwitz problem, see [Au]).

2.1 Mapping class groups

We first introduce the notion of a mapping class group. We note that all of the material in this section can be found in greater detail in Farb and Margalit's book [FM].

Definition 2.1. *The* mapping class group *of an oriented topological space X, denoted* $\mathrm{MCG}(X)$*, is defined to be*

$$\mathrm{MCG}(X) := \pi_0(\mathrm{Homeo}^+(X)) = \mathrm{Homeo}^+(X)/\sim$$

where $h \sim h'$ means h is isotopic to h'. Here $\mathrm{Homeo}^+(X)$ *denotes the group of orientation preserving homeomorphisms of X.*

If X has boundary, we define $MCG(X, \partial X)$ to be the group of isotopy classes of homeomorphisms that fix the boundary pointwise. Likewise, if X has punctures we define $MCG(X)$ to be the isotopy classes of homeomorphisms that permute the punctures among themselves.

In this paper we will be concerned with the case when $X = S$ is a surface. Let $\Gamma_S = \{\gamma \subset S : \gamma \text{ is a simple closed curve}\}/\sim$, where $\gamma_1 \sim \gamma_2 \iff \gamma_1$ is isotopic to γ_2. We say that a simple closed curve $\gamma \subset S$ is called *separating* if $S \backslash \gamma$ is not connected, and *nonseparating* otherwise.

Let $\mathcal{A} = I \times S^1$ denote the annulus with coordinates parametrized by $(r, t) \in [0, 1] \times [0, 2\pi]/(\{0\} \sim \{2\pi\})$.

Definition 2.2. *Let γ be a simple closed curve and let N be an annular neighborhood of γ, i.e. there is a homeomorphism $h : N \to \mathcal{A}$. A* Dehn twist *$\gamma \subset S$, denoted $T_\gamma \in \mathrm{Homeo}^+(S)$, is given by*

$$T_\gamma(x) := \begin{cases} h^{-1} \circ f \circ h(x) & x \in N \\ x & x \notin N \end{cases}$$

where $f \in \mathrm{Homeo}^+(\mathcal{A})$ is the twist given by

$$f(r, t) = (r, t + 2\pi r).$$

We will abuse notation and write T_γ to denote the isotopy class of the Dehn twist about $\gamma \in \Gamma_S$ in $\mathrm{MCG}(S)$. It is straightforward to check that twisting about an isotopy class of simple closed curves is well defined up to isotopy.

We call a Dehn twist *positive* if it twists the surface in a clockwise manner with respect to the surface orientation, and *negative* otherwise. An illustration of a positive Dehn twist is given in figure 1.

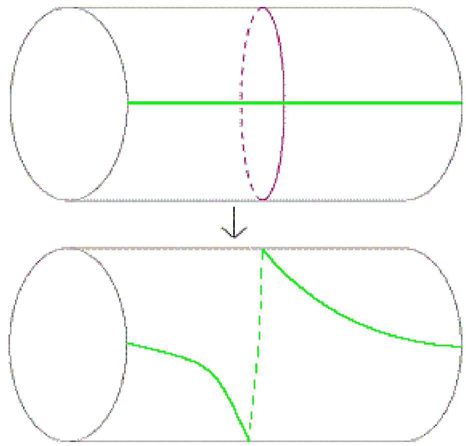

Figure 1: A positive Dehn twist in an annulus.

An important fact about mapping class groups of surfaces is that they are generated by Dehn twists. Let S_g denote a surface of genus g with no boundary.

Theorem 2.3 (Dehn-Lickorish)**.** *For $g \geq 0$, the group $\mathrm{MCG}(S_g)$ is generated by finitely many Dehn twists about nonseparating simple closed curves.*

Wee [FM] for a proof. In fact more is true: Any surface with boundary (but no punctures) is generated by Dehn twists. See section 5 for the Gervais presentation of the mapping class group which gives a presentation for surfaces with boundary.

Example Consider the case $g = 1$. Then the compact surface of genus g is the torus T^2, which has mapping class group

$$\mathrm{MCG}(T^2) = SL_2(\mathbb{Z}) = \langle a, b \mid cdc = dcd, (cd)^6 = 1 \rangle,$$

via the action on the two basic curves u and v in the picture.

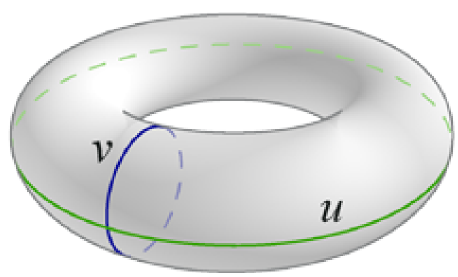

Figure 2: Torus with distinguished curves u, v (picture from [Ir]).

In a similar vein, the mapping class group of $S_{1,1}$, the torus with one hole, is the braid group on three strands

$$B_3 = \langle c, d \mid cdc = dcd \rangle.$$

Before we end the section, we list a few basic lemmas from [FM]

Lemma 2.4. *Let $f \in \mathrm{MCG}(S)$. Then we have $fT_\gamma f^{-1} = T_{f(\gamma)}$.*

Lemma 2.5. *Let $\gamma_1, \gamma_2 \in \Gamma_S$. Then $T_{\gamma_1} = T_{\gamma_2}$ if and only if $\gamma_1 = \gamma_2$.*

2.2 Lefschetz fibrations

We now introduce Lefschetz fibrations, how they relate to symplectic 4-manifolds, and how they can be associated with mapping class group factorizations.

Definition 2.6. *A* Lefschetz fibration *on an oriented, compact smooth 4-manifold X is a smooth surjection $\pi : X \to S$, where S is a compact, connected, oriented surface that is a submersion everywhere except at finitely many critical points where there are local complex coordinates that give π the form $\pi(z_1, z_2) = z_1^2 + z_2^2$.*

In this paper, we will only consider the cases $S = S^2$ or $S = D^2$. We will use Σ to denote the regular fiber of a Lefschetz fibration.

Lefschetz fibrations are of particular interest because they correspond to symplectic structures on 4-manifolds. We state two theorems that make this correspondence precise.

Theorem 2.7 ([Gom])**.** *Let $f : X \to S$ be a Lefschetz fibration and let $[\Sigma] \in H_2(X; \mathbb{R})$ denote the homology class of the fiber. If $[\Sigma] \neq 0$, then X admits a symplectic structure with symplectic fibers.*

Theorem 2.8 ([Don])**.** *Let (X, ω) be a compact symplectic 4-manifold. Then X carries a symplectic Lefschetz pencil, i.e. there exists a finite set $\Gamma \subset X$ and a map $f : X \backslash \Gamma \to \mathbb{CP}^1 = S^2$ such that f is modeled on $(z_1, z_2) \mapsto (z_1 : z_2)$ near each point of Γ, and f is a Lefschetz fibration with (noncompact) symplectic fibers outside of Γ.*

Corollary 2.9. *The manifold $\hat{X}$ obtained from X by blowing up at the points of Γ admits a Lefschetz fibration $f : \hat{X} \to S^2$ with symplectic fibers.*

As a result of this connection to symplectic topology, symplectic techniques can be used to prove statements about Lefschetz fibrations, and ultimately about mapping class groups of surfaces. More will be said on this in Section 3.

The local condition prescribes a specific topological structure to Lefschetz fibrations. Let $\Gamma \subset X$ denote the set of critical points and let $p \in \Gamma$. Let U be a neighborhood of p such that the fibration can be described by $\pi : (z_1, z_2) \mapsto z_1^2 + z_2^2$, where $(0, 0)$ corresponds to the point p in these local coordinates. Then the regular fibers in the restriction $\pi : U \to S$ are homeomorphic to $\{z_1^2 + z_2^2 = 1\} \cong TS^1 \cong \mathcal{A}$. Also note that the local condition can be rephrased as $\pi : (z_1, z_2) \mapsto z_1^2 + z_2^2 = (z_1 + iz_2)(z_1 - iz_2) = zw$ by a $\mathbb{C}$-linear change of coordinates. Thus the singular fiber in the restriction is given by $\{zw = 0\}$ is a transverse intersection of 2 open disks that intersect at the point $z = w = 0$.

More precisely, let $\gamma : I \to S$ be a path where $\gamma(t) \in S \backslash \pi(\Gamma)$ for $t < 1$ and $\gamma(1) = q \in \pi(\Gamma)$. As $t \to 1$ there is an open set homeomorphic to the annulus in the fiber $\pi^{-1}(\gamma(t))$ that gets pinched about a loop to a point until it locally becomes the wedge of two discs as described above. This loop in the annulus (a subset of the regular fiber) is called the *vanishing cycle* associated to the critical value q (Figure 3).

We now recall the definition of a fiber bundle.

Definition 2.10. *Let B be a connected topological space with a chosen base point $b_0 \in B$. A continuous map $p : E \to B$ is called a* fiber bundle *with fiber F if $p^{-1}(b_0) = F$ and p satisfies the local triviality condition, i.e. for each $b \in B$ there exists an open*

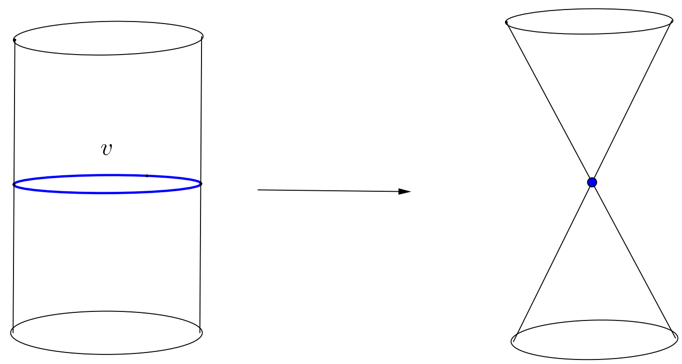

Figure 3: Vanishing cycle in an annulus.

neighbourhood U_b *of* b *and a homeomorphism* $\phi_b : p^{-1}(U_b) \to U_b \times F$ *such that the diagram commutes:*

$$\begin{tikzcd} p^{-1}(U_b) \arrow[rr, "\phi_b"] \arrow[dr, "p"'] & & U_b \times F \arrow[dl, "proj_1"] \\ & U_b & \end{tikzcd}$$

The following theorem of Ehresmann will be useful.

Theorem 2.11 ([Ehr])**.** *Any smooth map* $f : M \to N$ *between smooth, connected manifolds which is surjective, proper, and a submersion is a fiber bundle.*

Given a Lefschetz fibration $\pi : X \to S$ let Γ denote the set of critical points. Let D_p denote a neighborhood of $p \in \pi(\Gamma)$ homeomorphic to a disk with $D_p \cap D_{p'} = \varnothing$ for $p, p' \in \pi(\Gamma)$. Let $\hat{S} = S \setminus \bigcup_{p \in \pi(\Gamma)} D_p$ and $\hat{X} = \pi^{-1}(\hat{S})$. Note that both $\hat{\Sigma}$ and $\hat{X}$ are closed subsets of compact manifolds, and hence are both compact.

Lemma 2.12. *The map* $\pi : \hat{X} \to \hat{\Sigma}$ *is a fiber bundle.*

Proof. Since both $\hat{X}$ and $\hat{\Sigma}$ are compact, π is proper and furthermore it satisfies all of the conditions of Theorem 2.11. ■

Proposition 2.13. *There is a monodromy homomorphism* $\phi : \pi_1(\Sigma \setminus \pi(\Gamma)) \to \mathrm{MCG}(F)$.

Proof sketch. Let $\gamma \in \pi_1(\Sigma \setminus \pi(\Gamma))$. For each $p \in \pi(\Gamma)$, choose mutually disjoint disks $D_p \ni p$ such that $\gamma \cap D_p = \varnothing$, and as before set $\hat{\Sigma} = \Sigma \setminus \bigcup_{p \in \pi(\Gamma)} D_p$ and $\hat{X} = \pi^{-1}(\hat{\Sigma})$. By Lemma 2.12 we see that $\pi : \hat{X} \to \hat{\Sigma}$ is a fiber bundle.

For each $x \in \gamma$, choose an open neighborhood $U_x \ni x$ that satisfies the locally trivial fiber bundle condition. Since $\gamma \subset \hat{\Sigma}$ is compact, there is a finite subcover $\{U_i\}_{i \leq n} \subset \{U_x\}_{x \in \gamma}$, ordered along γ. Let $\psi_{i,i+1} = \phi_{i+1}^{-1} \circ \phi_i : \pi^{-1}(U_i \cap U_{i+1}) \to \pi^{-1}(U_i \cap U_{i+1})$ denote the transition map. Define $\psi := \psi_{n-1,n} \circ \cdots \circ \psi_{1,2} \in \mathit{Homeo}(U_n \cap U_1)$. It

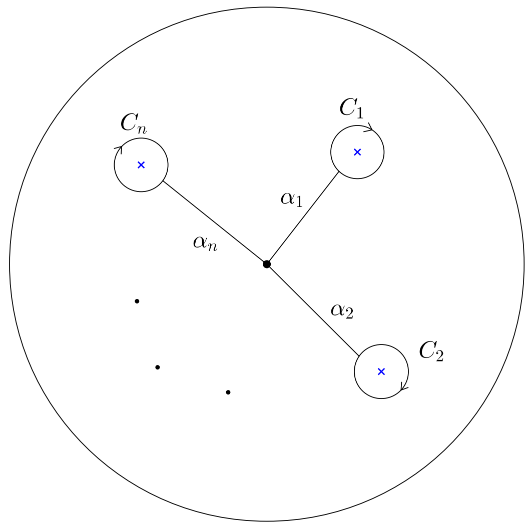

Figure 4: A good generating set for $\pi_1(\mathbb{D}\backslash\pi(\Gamma))$.

can be shown that ψ does not depend on the choice of cover and that homotopic curves give rise to isotopic homeomorphisms. We define $\phi(\gamma) = \psi$. Finally, it is clear that ϕ preserves the group structure since concatenating loops amounts to composing homeomorphisms. ■

2.3 Fibrations from monodromy data

Proposition 2.13 says that a Lefschetz fibration gives rise to a monodromy homomorphism. This section will outline how the converse is also true: given a monodromy homomorphism we can construct a Lefschetz fibration with the prescribed monodromy data. We will assume throughout that $S = D$, i.e. the base space is a closed disk.

Choose a basepoint $x_0 \in D\backslash\pi(\Gamma)$. For each critical value $p \in \pi(\Gamma)$, choose mutually disjoint neighborhoods D_p homeomorphic to a disc, and let $C_p = \partial D_p$. Let $\alpha_p : I \to S$ be a path with $\alpha_p(0) = x_0$ and $\alpha_p(1) = p$.

Note that $D\backslash\pi(\Gamma) = D_n$, the disk with n holes. Let $\gamma_p \in \pi_1(D_n)$ be the closed curve based at x_0 that follows α until intersecting C_p, loops around the circle C_p and follows α back to x_0 (see Figure 4). Then $\{\gamma_p\}_{p\in\pi(\Gamma)}$ is a generating set for $\pi_1(D_n)$ such that each γ_p encloses exactly one critical value p. We will call such generating sets, for instance the set in Figure 4, *good*.

Proposition 2.14. *If γ_p is chosen as above, $\phi(\gamma_p) = T_{v_p}$, i.e. the monodromy around γ_p is a Dehn twist about the vanishing cycle v_p associated to p.*

For a proof, see [Cas].

Theorem 2.15. *There is a 1-1 correspondence between the set of monodromy representations $\{\phi : \pi_1(D_n) \to \mathrm{MCG}(F)\}$, and the set of Lefschetz fibrations $\pi : X \to D$ admitting fiber F with n vanishing cycles.*

To see how to construct a Lefschetz fibration from a monodromy $\phi : \pi_1(D_n) \to \mathrm{MCG}(F)$, first choose a good generating set $\{\gamma_p\}$ for $\pi_1(D_n)$. From the monodromy

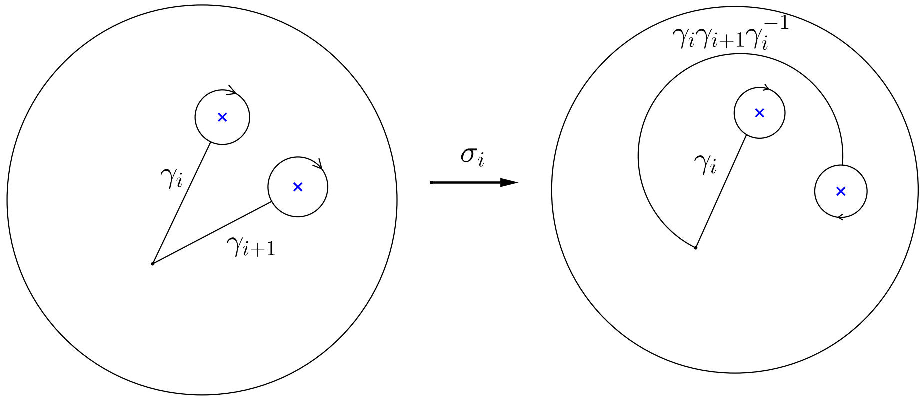

Figure 5: A Hurwitz move on a pair of good generators.

we have that $\phi(\gamma_p) = T_{v_p}$, where the simple closed curve v_p is well defined up to isotopy due to Lemma 2.5. The following sketch of a construction will make the collection $\{v_p\}$ the vanishing cycles of the resulting Lefschetz fibration.

Start with the trivial bundle $\Sigma \times D^2$. Given an ordering on the vanishing cycles, there are unique lifts $v_p \hookrightarrow \Sigma \times S^1 \subset \partial(\Sigma \times D^2)$ such that when we attach 2-handles along these embeddings with a prescribed framing condition, the resulting 4-manifold is a Lefschetz fibration with the given vanishing cycles. Intuitively, a vanishing cycle corresponds to an element of $H_1(X)$ being deleted, which corresponds to attaching a 2-handle in the construction which makes the vanishing cycle the boundary of a 2-cell. For further details, see [Ful].

The ambiguity in this construction comes from the choice of generators for $\pi_1(D_n)$. The issue is that, *a priori*, a different choice of generators might result in a different Lefschetz fibration. However, it turns out that this does not occur.

Proposition 2.16. *Any other set of good generators for $\pi_1(D\backslash\pi(\Gamma))$ can be obtained by the action of an element of $\mathrm{MCG}(D\backslash\pi(\Gamma)) = \mathrm{MCG}(D_n) \cong B_n$, the braid group on n strands.*

The generators for $\mathrm{MCG}(D_n)$ are given by "half twists" σ_i that switch the punctures i and $i+1$.

We can describe the action of σ_i on $\pi_1(D_n)$ by observing the action on the ordered tuple of generators

$$(\gamma_1, \ldots, \gamma_i, \gamma_{i+1}, \ldots, \gamma_n) \mapsto (\gamma_1, \ldots, \gamma_i \cdot \gamma_{i+1} \cdot \gamma_i^{-1}, \gamma_i, \ldots, \gamma_n).$$

An illustration of this is given in Figure 5.

Composing with the monodromy homomorphism gives an action of B_n on the corresponding Dehn twists $(T_{v_1}, \ldots, T_{v_n})$ given by

$$(T_{v_1}, \ldots, T_{v_i}, T_{v_{i+1}}, \ldots, T_{v_n}) \mapsto (T_{v_1}, \ldots, T_{v_i} T_{v_{i+1}} T_{v_i}^{-1}, T_{v_i}, \ldots, T_{v_n}),$$

which is the i^{th} *Hurwitz move.*

There is one additional action which preserves the structure of the Lefschetz fibration, which is the action of an element of the mapping class group $\varphi \in \mathrm{MCG}(\Sigma)$ on the Lefschetz fibration X by fiber-wise action. This action maps each vanishing cycle v to an image cycle $\varphi(v)$, and thus acts as conjugation on Dehn twists in $\mathrm{MCG}(F)$, giving rise to the global conjugation action

$$(T_{v_1}, \dots, T_{v_n}) \mapsto (\varphi T_{v_1} \varphi^{-1}, \dots, \varphi T_{v_n} \varphi^{-1})$$

Combining these, we state:

Proposition 2.17. *Suppose $\{\gamma_i\}$ and $\{\tilde{\gamma}_i\}$ are good generating sets of $\pi_1(T_n)$ that are in the same orbit under the actions of the Hurwitz moves and global conjugation. Then the Lefschetz fibrations constructed from each generating set are diffeomorphic.*

Finally, we prove a proposition for future reference.

Proposition 2.18. *Let $\pi : X \to D$ be a Lefschetz fibration with n singular fibers and regular fiber Σ. Then $\chi(X) = \chi(\Sigma) + n$.*

Proof. The Lefschetz fibration is constructed by taking the trivial product $\Sigma \times D^2$ and attaching 2-handles to a choice of n vanishing cycles. The trivial product deformation retracts to Σ so we have $\chi(\Sigma \times D^2) = \chi(F)$. Attaching the 2-handles corresponds to adding 2-cells for every vanishing cycle, giving $\chi(X) = \chi(F) + n$. ■

2.4 The Hurwitz problem

From the previous sections, we know that Lefschetz fibrations with regular fiber Σ are classified, up to isomorphism, by an ordered tuple of Dehn twists (T_{γ_i}) corresponding to vanishing cycles γ_i on Σ, up to the actions of two moves:

1. global conjugation $(T_{\gamma_i}) \mapsto (\varphi T_{\gamma_i} \varphi^{-1}) = (T_{\varphi(\gamma_i)})$ for $\varphi \in \mathrm{MCG}(\Sigma)$; and
2. Hurwitz moves

$$(T_{\gamma_1}, \dots, T_{\gamma_i}, T_{\gamma_{i+1}}, \dots, T_{\gamma_n}) \mapsto (T_{\gamma_1}, \dots, T_{\gamma_i} T_{\gamma_{i+1}} T_{\gamma_i}^{-1}, T_{\gamma_i}, \dots, T_{\gamma_n}).$$

Thus, we can state the Hurwitz problem for the mapping class group, which asks if we can understand the equivalence classes of ordered tuples (T_{γ_i}) under these actions. Equivalently, is there a method to determine if two such tuples are related by a series of global conjugations and Hurwitz moves?

First, we do an example to show that this problem could be tractable.

Example Consider the case where the fiber Σ is the torus, and we have two vanishing cycles. We first note that a simple closed curve on the torus, which determines a unique corresponding Dehn twist (Lemma 2.5), is itself determined by a pair of coprime positive integers (a, b) corresponding to the curve's intersection numbers with the meridian and longitude of the torus. Thus, a curve can be written as an column vector in $\mathbb{Z}^2$, so a pair of curves can be written as a matrix

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}).$$

Then, since elements of $\mathrm{MCG}(T^2) = \mathrm{SL}_2(\mathbb{Z})$ act as left-multiplication on curves, their action via global conjugation on the pair of twists is also left-multiplication on this matrix. In addition, one can calculate the action of a Hurwitz move to be

$$A \mapsto \begin{pmatrix} c - \det\{A\} \cdot a & a \\ d - \det\{A\} \cdot b & b \end{pmatrix}.$$

To determine the possible equivalence classes of pairs of curves under these two actions, we first note that a matrix A with a, b coprime can be mapped to a matrix of the form $\begin{pmatrix} 1 & m \\ 0 & n \end{pmatrix}$ via a left-multiplication taking $(a\ b)^{\mathrm{T}}$ to $(1\ 0)^{\mathrm{T}}$. Thus, each equivalence class corresponds to a pair (m, n).

Now, a left-multiplication preserving the first column $(1\ 0)^{\mathrm{T}}$ must be of the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2\mathbb{Z}$ and thus map the class $[(m, n)]$ to the class $[(m + kn, n)]$. Thus, m is determined only up to a factor of n, so we can replace m by its residue $\overline{m} \in \mathbb{Z}/n\mathbb{Z}$.

Finally, a Hurwitz move takes the class $[(\overline{m}, n)]$ to the class $[(\overline{m}^{-1}, -n)] = [(-\overline{m}^{-1}, n)]$, as for $\overline{k} = \overline{m}^{-1}$, we get $mk = 1 + ni$ and

$$\begin{pmatrix} k & k - i \\ -n & m - n \end{pmatrix} \begin{pmatrix} m - n & 1 \\ n & 0 \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & -n \end{pmatrix}.$$

Thus, the desired equivalence classes in this case are characterized by pairs (m, n), where $m \in \mathbb{Z}/n\mathbb{Z}$ and two pairs (m, n), (k, n') are equivalent if $n = n'$ and $mk = -1$ in $\mathbb{Z}/n\mathbb{Z}$.

We can extend this method to $k > 3$ vanishing cycles, with similar results at the level of global conjugation. However, the Hurwitz moves become an action of $B_k/Z(B_k)$ on the resulting classes, instead of $\mathbb{Z}/2\mathbb{Z}$ as they are for two vanishing cycles, making the study more difficult. Even in the case $k = 3$ for the torus, the Hurwitz action makes a huge number of classes equivalent, and we could not discern a pattern in this collapse (though we did notice an interesting Fibonacci-like growth pattern in the binary tree generated by the two Hurwitz moves).

We can also extend this method to surfaces of higher genus, with 2 vanishing cycles. Here, the single Hurwitz move σ_1 is an involution up to global conjugation, as noted above:

$$\begin{aligned} \sigma_1^2(T_\gamma, T_{\gamma'}) &= \sigma_1(T_\gamma T_{\gamma'} T_\gamma^{-1}, T_\gamma) \\ &= (T_\gamma T_{\gamma'} T_\gamma T_{\gamma'}^{-1} T_\gamma^{-1}, T_\gamma T_{\gamma'} T_\gamma^{-1}) \\ &= T_\gamma T_{\gamma'} (T_\gamma, T_{\gamma'}) (T_\gamma T_{\gamma'})^{-1}. \end{aligned}$$

Thus, it suffices to classify pairs of curves up to the action of the MCG. In fact, we give a stronger, inductive classification of k-tuples of curves up to this action, where the base case $k = 0$ is trivial.

Suppose we are given two k-tuples of curves T_{γ_i} and $T_{\gamma_i'}$. First, we check if the initial $k-1$-tuples of curves are equivalent up to action of the MCG. If not, then the extended k-tuples are clearly not equivalent. If so, then we can assume without loss of generality that $\gamma_i' = \gamma_i$ for $i < k$, so that we are looking at the question of whether $(T_{\gamma_1}, \ldots, T_{\gamma_{k-1}}, T_{\gamma_k})$ and $(T_{\gamma_1}, \ldots, T_{\gamma_{k-1}}, T_{\gamma_k'})$ are equivalent.

Here, we use the change-of-coordinates principle. This states that such k-tuples are equivalent if and only if the surfaces obtained by cutting along all of the curves are diffeomorphic, and the resulting attaching data on the boundary coincide. As the first $k-1$ curves are the same, we can pay attention only to the cut along the last curve, which gives a small amount of boundary attaching data to keep track of.

Since both of these invariants are ultimately discrete (the first determined by the genus and number of boundary components of each component surface obtained, the second determined by colorings on circular graphs), this gives an algorithmic classification of k-tuples of curves up to global conjugation.

3 Results from symplectic geometry

3.1 Boundaries

We have a number of results about Lefschetz fibrations whose fibers are surfaces $\bar{\Sigma}$ with nontrivial boundary, using the machinery of symplectic geometry. In particular, we state Corollary 2.21 in [Ent]:

Theorem 3.1. *If a 4-dimensional manifold M with a Lefschetz fibration structure has boundary isomorphic to that of $\bar{\Sigma} \times D^2$ (with contact structures), then M must be diffeomorphic to $\bar{\Sigma} \times D^2$.*

Now, recall that the boundary ∂M of such a Lefschetz fibration is determined by the boundary monodromy $T_{\gamma_1} \dots T_{\gamma_n}$ on the fiber $\bar{\Sigma}$, given by the vanishing cycles γ_i. Thus, we have the following result as a consequence.

Theorem 3.2. *There is no nontrivial factorization of the identity on $\bar{\Sigma}$ into positive Dehn twists.*

Proof. Suppose there is such a factorization $\mathrm{id}_{\bar{\Sigma}} = T_{\gamma_1} \dots T_{\gamma_n}$ for some curves γ_i. Then we can construct a Lefschetz fibration $M \to D^2$ with fiber $\bar{\Sigma}$ and vanishing cycles γ_i.

Since M has trivial boundary monodromy by construction, the preceding observation gives $\partial M = \partial(\bar{\Sigma} \times D^2)$. By 3.1, we conclude that $M \cong \bar{\Sigma} \times D^2$. But by Proposition 2.18, this implies

$$\chi(\bar{\Sigma}) + n = \chi(M) = \chi(\bar{\Sigma} \times D^2) = \chi(\bar{\Sigma}),$$

so $n = 0$ and the factorization is trivial. ■

We can obtain further results with the concept of stabilization.

Definition 3.3. *The* (positive) stabilization *of a Lefschetz fibration with fiber $\bar{\Sigma}$ a surface with boundary is the Lefschetz fibration with*

1. *fiber $\bar{\Sigma}' = \bar{\Sigma} \cup 1$-handle and*
2. *an additional vanishing cycle γ in $\bar{\Sigma}'$, appended to the end of the list of vanishing cycles, that intersects the co-core c of the 1-handle once (and thus passes through the handle exactly once).*

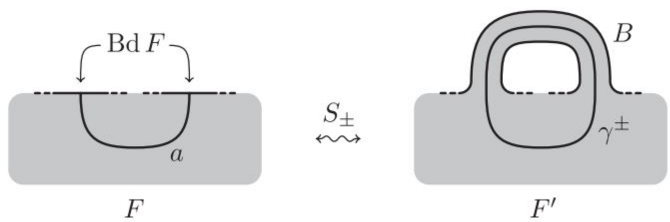

Figure 6: The process of stabilizing F to F' (picture taken from [Zu]).

We can define a similar *negative* stabilization using the negative Dehn twist T_γ^{-1} for the new vanishing cycle γ, but this does not give a Lefschetz fibration structure on the resulting 4-manifold.

Note that positive stabilization preserves the total space M of the Lefschetz fibration, since the additional handle in the fiber cancels with the 2-handle attached via the additional vanishing cycle. Thus, we have the following analogue of 3.2.

Theorem 3.4. *Suppose $\gamma_1, \dots, \gamma_n$ are curves obtained from stabilizing the surface $\bar{\Sigma}$ n times. Then a factorization $T_{\gamma_1} \dots T_{\gamma_n} = T_{\delta_1} \dots T_{\delta_m}$ must have $m = n$.*

Proof. We can stabilize the trivial Lefschetz fibration structure on $\bar{\Sigma} \times D^2$ n times to obtain a Lefschetz fibration on $\bar{\Sigma} \times D^2$ with vanishing cycles γ_i.

Then we construct a Lefschetz fibration M with vanishing cycles δ_m. This gives two fibrations with the same boundary monodromy, so $\partial M = \partial(\bar{\Sigma} \times D^2)$ and 3.1 gives $M \cong \bar{\Sigma} \times D^2$.

Since both of these manifolds are Lefschetz fibrations over the disk, with fiber $\bar{\Sigma}'^{\cdots\prime}$ obtained from $\bar{\Sigma}$ after n stabilizations, a calculation of Euler characteristic using Proposition 2.18 gives

$$\chi(\bar{\Sigma}') + m = \chi(M) = \chi(\bar{\Sigma} \times D^2) = \chi(\bar{\Sigma}') + n$$

and $m = n$. ∎

3.2 Overtwisted structures on contact 3-manifolds

The boundary of a Lefschetz fibration is a contact 3-manifold. We call a contact 3-manifold fillable if it can be written as the boundary of some Lefschetz fibration.

All contact 3-manifolds fall into two disjoint classes, tight and overtwisted (for details, see [El]). Moreover, we have Theorem 1.1 of [Pr]:

Theorem 3.5. *Every overtwisted manifold is non-fillable.*

In particular, the boundary of a Lefschetz fibration is a tight contact 3-manifold.

We also know that if we negatively stabilize a contact 3-manifold, then the 3-manifold becomes overtwisted (remark 9.2.12 of [OS]), and it remains overtwisted even if we positively stabilize it again. These facts combine to give the following consequence, on mapping class group factorizations:

Theorem 3.6. *Suppose that $\phi = \bar{\phi} T_\delta^{-1} T_{\gamma_1} \dots T_{\gamma_n}$ is the monodromy obtained, from the surface $\bar{\Sigma}$ with monodromy $\bar{\phi}$, by a negative stabilization $T_{\gamma_i}^{-1}$ and n positive stabilizations T_i. Then ϕ cannot be factorized into a product of positive Dehn twists.*

Proof. Let ϕ be constructed with at least one negative stabilization, with corresponding contact 3-manifold N, and suppose F is a factorization of ϕ into a product of positive Dehn twists. Then by Proposition 2.17, we can construct a Lefschetz fibration $M \to D^2$ with monodromy representation given by the factors of F, which would have boundary monodromy $F = \phi$.

As the boundary of a Lefschetz fibration over D^2 is a fiber bundle over S^1, and is thus determined by the boundary monodromy, we obtain $\partial M = N$, giving a symplectic filling of N. But N is overtwisted, by the remark made before the proof, which contradicts Theorem 3.5. Thus no such factorization F can exist. ■

In the next few sections, we wish to shed light on the preceding results using techniques related to the structure of the MCG, instead of the previously-referenced techniques in symplectic geometry.

4 Investigation into planar surfaces

In the previous section, we stated three theorems about mapping class group factorizations, namely theorems 3.2, 3.4, and 3.6, which come from symplectic geometry. In this section, we will prove them for a genus 0 surface. In order that our surface does not gain any genus, we only allow stabilizations of the disk that add boundary components, which forms a disk with n holes D_n.

We choose D_n as a surface of study because $\mathrm{MCG}(D_n)$ is easy to understand: a presentation of $\mathrm{MCG}(D_n)$ is given by Margalit and McCammond [MM], as follows. Let us call a simple closed curve in D_n *convex* if it is isotopic to the boundary of the convex hull of a collection of holes A, and a Dehn twist convex if it twists about such a convex curve. We denote such a Dehn twist by T_A, where A is the collection of holes enclosed by the corresponding curve (a mild abuse of notation). Then $\mathrm{MCG}(D_n)$ has a presentation with all possible T_A as generators, and with the following two classes of relations:

- Dehn twists around disjoint curves commute.
- The generalized lantern relation, which states that

  $$T_{A\cup B\cup C}T_AT_BT_C = T_{A\cup B}T_{B\cup C}T_{C\cup A},$$

  where A, B, C are disjoint collections of holes (see Figure 7).

From this presentation of $\mathrm{MCG}(D_n)$, Plamenevskaya and Van Horn-Morris [PV] found a natural invariant on $\mathrm{MCG}(D_n)$, which arises from two key observations.

The first observation is that a Dehn twist enclosing many holes can be decomposed, using the lantern relation, into Dehn twists that enclose fewer holes; for instance, we can write

$$T_{A\cup B\cup C} = T_{A\cup B}T_{B\cup C}T_{C\cup A}{T_A}^{-1}{T_B}^{-1}{T_C}^{-1}$$

. By repeatedly applying the lantern relation, we can express any Dehn twist, or indeed any element of $\mathrm{MCG}(D_n)$, as a product of Dehn twists that enclose either one or two holes. This representation is unique once we take images in the abelianization of $\mathrm{MCG}(D_n)$, written as $\mathrm{MCG}^{\mathrm{ab}}(D_n)$.

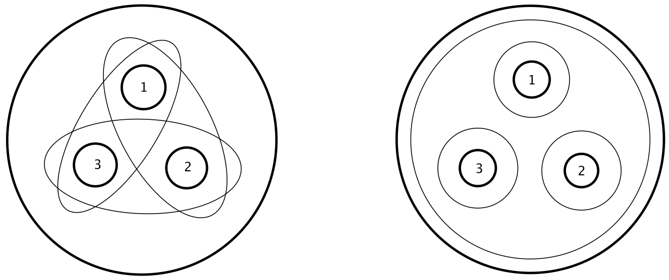

Figure 7: The curves represented on the two sides of the lantern relation.

The second observation is that the set of holes which a curve encloses determines the orbit of the curve under the action of $\mathrm{MCG}(D_n)$, so since conjugacy classes are collapsed under abelianization, the set of holes A enclosed by a Dehn twist determines its image in $\mathrm{MCG}^{\mathrm{ab}}(D_n)$.

Thus, if we label the holes in D_n from 1 to n, we know from the two observations that an element in $\mathrm{MCG}^{\mathrm{ab}}(D_n)$ breaks down into m_i copies of $T_{\{i\}}$ and m_{ij} copies of $T_{\{i,j\}}$. These invariants $\{m_{ij}, m_i\}$ determine the element uniquely (so that $\mathrm{MCG}^{\mathrm{ab}}(D_n) = \mathbb{Z}^{\binom{n}{2}}$).

Let us consider $\{m_{ij}, m_i\}$ for specific examples of $\mathrm{MCG}^{\mathrm{ab}}(D_n)$. We will call a Dehn twist a boundary Dehn twist if it encloses a single hole, that is, it is $T_{\{i\}}$ for some i. In this case, we observe that $m_k(T_i) = \delta_{ik}$ is 0 unless $i = k$, and $m_{ij}(T_k) = 0$ for all i, j, k. For a non-boundary Dehn twist that encloses more than one hole, the result is less obvious:

Proposition 4.1. *For a positive non-boundary Dehn twist T_γ enclosing r holes, its $\{m_{ij}, m_i\}$ is given as follows:*

1. *$m_{kl} = 1$ if holes k and l are both enclosed in T_γ, and 0 otherwise;*
2. *$m_k = -(r-2)$ if the hole k is enclosed in T_γ, and 0 otherwise.*

The proposition can be easily proven by induction and repeated application of the lantern relation; for details, see [PV]. We remark that only non-boundary Dehn twists contribute positively to m_{ij}.

In addition, Plamenevskaya and Van Horn-Morris proved a useful lemma [PV], which we have modified to suit our purposes.

Lemma 4.2. *Let $\phi \in \mathrm{MCG}(D_n)$ be a product of positive Dehn twists, and suppose a hole k is enclosed by p boundary Dehn twists and q non-boundary Dehn twists in ϕ. Suppose F is a factorization of ϕ into positive Dehn twists. Then there are no more that $p+q$ non-boundary Dehn twists in F that enclose k.*

Proof. Suppose that F contains l non-boundary Dehn twists that each encloses k, then we want to prove that $l \leq p + q$.

Consider the q non-boundary Dehn twist which are in ϕ and enclose k, and suppose they enclose $a_1, ..., a_n \geq 2$ holes respectively. Suppose that F contains l non-boundary Dehn twists that enclose k, which enclose $b_1, ..., b_n \geq 2$ holes respectively.

First we will calculate $\sum_i m_{ik}$ for ϕ and F, where i is some hole in D_n. By Proposition 4.1, every pair of holes enclosed by a non-boundary Dehn twist contributes contributes a $+1$ to our sum. Thus, we have

$$\sum_i m_{ik}(\phi) = (a_1 - 1) + (a_2 - 1) + ... + (a_q - 1).$$

Similarly,

$$\sum_i m_{ik}(F) = (b_1 - 1) + (b_2 - 1) + ... + (b_l - 1),$$

and because F is a factorization of ϕ, we have

$$\sum_i m_{ik}(\phi) = \sum_i m_{ik}(F).$$

Now we calculate m_k for ϕ and F. By Proposition 4.1, we have

$$m_k(\phi) = p - (a_1 - 2) - (a_2 - 2) - ... - (a_q - 2),$$

while

$$m_k(F) = m_k(\phi) \geq -(b_1 - 2) - (b_2 - 2) - ... - (b_l - 2),$$

where the p in the first equation comes from the p boundary Dehn twists in ϕ, and the inequality is because F may contain boundary Dehn twists about k, which have not been considered and would increase m_k.

Thus we have the following relations:

$$a_1 + a_2 + ... + a_q - q = b_1 + b_2 + ... + b_n - l, \tag{1}$$
$$p - (a_1 + a_2 + ... + a_q - q) \geq -(b_1 + b_2 + ... + b_n - 2l). \tag{2}$$

Substituting (1) into (2), we get the desired inequality $l \leq p + q$. ■

We are now ready to prove a few preliminary results:

Proposition 4.3. *There is no factorization of the identity in* $\mathrm{MCG}^(D_n)$ *into a positive number of positive Dehn twists.*

Note that this is a special case of Theorem 3.2, the general theorem pertaining to factorizations of the identity. In fact, we prove the theorem by passing to $\mathrm{MCG}^{\mathrm{ab}}(D_n)$ via the abelianization map, so the theorem is true there as well.

Proof. Suppose we have some factorization F of id consisting of positive Dehn twists. We will show that F contains no non-boundary Dehn twists, and then show it contains no boundary Dehn twists, which completes the proof.

By Lemma 4.2 in the case $p, q = 0$, for any hole k there are no more than zero non-boundary Dehn twists that enclose k. Thus F contains no non-boundary Dehn twists.

Thus only boundary Dehn twists can contribute to $\{m_{ij}, m_i\}$ of F. Recall that a boundary Dehn twist enclosing some hole k contributes a $+1$ to m_k. But all $m_i = 0$ for id, so F does not contain any boundary Dehn twists either, and we are done. ■

Proposition 4.4. *If we have a non-boundary Dehn twist T_γ about some simple closed curve γ, and F is a factorization of T_γ into positive Dehn twists, then F must consist of the single term T_γ.*

Proposition 4.4 remains true if we allow T_γ to be a boundary Dehn twist; the proof in this case is similar (and in fact simpler).

Proof. To prove this statement, we first show that F contains precisely one non-boundary Dehn twist, and that Dehn twist encloses the same holes as T_γ. Then, if there are remaining boundary Dehn twists in F, they will form a factorization of id once we pass to $\mathrm{MCG}^{\mathrm{ab}}(D_n)$, contradicting Proposition 4.3. This gives F consists of a single Dehn twist, which must be T_γ by Lemma 2.5.

We consider some hole k enclosed in T_γ. By Lemma 4.2, k is enclosed by no more than one non-boundary Dehn twist in F. Suppose k is not enclosed by any non-boundary Dehn twists. Then $m_{k'k}(F) = 0$, for all other k' enclosed by T_γ. However, $m_{k'k}(T_\gamma) = 1$ by Proposition 4.1, giving a contradiction. So k must be enclosed by one and only one non-boundary Dehn twist in F; we label that Dehn twists $T_{\gamma'}$.

Now we show that $T_{\gamma'}$ encloses the same holes as T_γ. A hole l not enclosed by T_γ cannot be enclosed by $T_{\gamma'}$, since $m_{kl} = 0$, so l and k cannot be enclosed by the same non-boundary Dehn twist. Conversely, a hole k' enclosed by T_γ is also enclosed by $T_{\gamma'}$, since $m_{k'k}$=1 for F, so k' must be enclosed by the unique non-boundary Dehn twist T'_γ in F enclosing k. This finishes the proof. ■

Proposition 4.5. *Let $\phi \in \mathrm{MCG}(D_n)$ and $\phi = T_\alpha T_\beta$, where T_α and T_β are both non-boundary Dehn twists. If F is a factorization of ϕ in positive Dehn twists, then F consists only of two non-boundary Dehn twists $T_{\alpha'}$ and $T_{\beta'}$, which enclose the same holes as T_α and T_β respectively.*

As in Proposition 4.4, Proposition 4.5 remains true if we allow boundary Dehn twists.

Proof. We first show that F contains twists $T_{\alpha'}$ and $T_{\beta'}$ which enclose the same holes as T_α and T_β respectively. As in the proof of Lemma 4.4, we pass to the abelianization, where the strenghtening of Proposition 4.3 tells us that F cannot contain further Dehn twists, completing the proof.

We need to check four different cases, which arise from different arrangements of T_α and T_β. The four cases are:

1. α, β disjoint, so they enclose different sets of holes;
2. α, β each enclosing some holes that are not enclosed by the other, but the two curves sharing some holes;
3. β containing all holes contained in α, but not vice versa;
4. α, β enclosing exactly the same holes.

Case 1: α, β disjoint. Then we can treat T_α and T_β separately. Following the same kind of argument found in Proposition 4.4, we can conclude that F contains the desired $T_{\alpha'}$ and $T_{\beta'}$.

Case 2: α, β share some but not all holes that they enclose. First we consider any hole a which is enclosed by T_α but not by T_β. By Lemma 4.2, a is enclosed by at most one non-boundary Dehn twist in F. Because $m_{ia} = 1$ for some i also enclosed by T_α, we know that a must be enclosed by exactly one non-boundary Dehn twist in F, which we write as $T_{\alpha'}$. Now let us consider any other hole a' enclosed by T_α. Because $m_{a'a}(T_\alpha) = 1$, there must exist a non-boundary Dehn twist in F that encloses both a' and a. But there is only one non-boundary Dehn twist that encloses a, namely $T_{\alpha'}$, so we know that a' is enclosed by $T_{\alpha'}$. Thus $T_{\alpha'}$ encloses the same holes as T_α.

Similarly, by considering any hole b which is enclosed by T_β but not T_α and using the same arguments, we know that F contains exactly one non-boundary Dehn twist that encloses b, which we call $T_{\beta'}$. Moreover, $T_{\beta'}$ encloses the same holes as T_β.

Case 3: All the holes of β are enclosed in α, but not conversely. First we consider any hole a enclosed by T_α but not by T_β. We argue as we have done for case 2 and conclude that F contains exactly one non-boundary Dehn twist that encloses a, which we call $T_{\alpha'}$, and $T_{\alpha'}$ encloses the same holes as T_α.

Now consider some hole b that is enclosed by T_β (so b is also enclosed by T_α). By Proposition 4.2, b is enclosed by at most two non-boundary Dehn twists in F. Because $m_{b'b} = 2$ for any other b' in T_β, b must be enclosed by exactly two non-boundary Dehn twists in F. We know b is enclosed in $T_{\alpha'}$ already, so there must be exactly one other non-boundary Dehn twist $T_{\beta'}$ which encloses b.

We consider some other hole b' in T_β. Because $m_{b'b} = 2$, there must exist two non-boundary Dehn twists in F that encloses both k' and k. But the only two non-boundary Dehn twists that encloses b are $T_{\alpha'}$ and $T_{\beta'}$, so b' is enclosed by $T_{\beta'}$. Thus $T_{\beta'}$ encloses the same holes as T_β.

Case 4: α, β enclose exactly the same holes. We consider any two holes a and a' enclosed by both T_α and T_β. By Proposition 4.2, a and a' are enclosed by at most two non-boundary Dehn twists. Because $m_{a'a} = 2$, a and a' must both be enclosed by exactly two non-boundary Dehn twists, and they must be enclosed by the same Dehn twists. Thus F contains exactly two non-boundary Dehn twists that enclose the same holes as T_α and T_β, which we denote $T_{\alpha'}$ and $T_{\beta'}$.

We have checked all four cases and concluded that F contains $T_{\alpha'}$ and $T_{\beta'}$, which enclose the same holes as T_α and T_β respectively. The remainder of F is equivalent to id in $\mathrm{MCG}^{\mathrm{ab}}(D_n)$, which is trivial according to Proposition 4.3, so we are done. ■

So far, we have proven that on D_n, id and a single Dehn twist has no nontrivial factorization, and a product of two Dehn twists has only factorizations into two Dehn twists enclosing the same holes. It is not the case, however, that any factorization of a product of n Dehn twists for all n must consist of n Dehn twists. A counter example is provided by the lantern relation, $T_{A\cup B\cup C}T_AT_BT_C = T_{A\cup B}T_{B\cup C}T_{C\cup A}$, where a product of four Dehn twists is factorized by a product of three Dehn twists.

At the same time, by the theorem on 4-manifolds we know that the product of n Dehn twists that arise from stabilization can only be factorized into n Dehn twists. What, then, makes n Dehn twists which come from stabilization special, when compared to arbitrary Dehn twists? The insight is provided by our earlier counterexample. The reason the four Dehn twists could be the product of another three is that they satisfied the lantern relation, which changes the number of positive Dehn twists in a product. So the n Dehn twists formed from stabilization must be special because they do not satisfy the lantern relation; indeed, if they did we would be able to find a factorization of them with a different length. There is an intuitive way to explain the lack of lantern relations among Dehn twists formed from stabilization: whenever we add a Dehn twist by stabilization, the new Dehn twist encloses a hole which no other Dehn twist encloses; yet for some Dehn twists to satisfy the lantern relationship, the sets of holes involved must all be enclosed by two distinct Dehn twists. We formalize this intuition in the lemma below:

Lemma 4.6. *Let $\phi \in \mathrm{MCG}(D_n)$ be a product of positive Dehn twist. Say $T_{\gamma_k} \in \phi$, where T_{γ_k} is the only Dehn twist in ϕ that enloses a hole labeled k. Then, if F is a factorization of ϕ into positive Dehn twist, there exists exactly one Dehn twist in F that encloses k, which we label $T_{\bar{\gamma}_k}$. Moreover, $T_{\bar{\gamma}_k}$ encloses the same holes as T_{γ_k}.*

In other words, this lemma tells us that if a Dehn twist encloses some holes that no other Dehn twists do, then it is not involved in any relations and must appear in the factorization.

Proof. The proof is similar to our earlier proofs for factorizations of one and two Dehn twists. We show that there is only one Dehn twist $T_{\bar{\gamma}_k}$ in F enclosing k; then we show $T_{\bar{\gamma}_k}$ encloses the same holes as T_{γ_k}. We will check two separate cases: when T_{γ_k} is a boundary Dehn twist, and when it is a non-boundary Dehn twist.

First, we assume T_{γ_k} is a boundary Dehn twist. By Lemma 4.2, we know that F contains no non-boundary Dehn twist that encloses k. Because $m_k = 1$, F must contain exactly one boundary Dehn twist that encloses k, and that boundary Dehn twist satisfies the desired properties of $T_{\bar{\gamma}_k}$, so we are done.

The other possibility is when T_{γ_k} is a non-boundary Dehn twist. By Lemma 4.2, we know that F contains at most one non-boundary Dehn twist that encloses k. Assume the F contains zero non-boundary Dehn twists that encloses k, so all $m_{kj} = 0$. However, T_{γ_k} is a non-boundary Dehn twist, so we must have some $m_{kj} = 1$, where j is also enclosed by T_{γ_k}, which gives a contradiction. Thus F contains exactly one non-boundary Dehn twist that encloses k, and we will label it $T_{\bar{\gamma}_k}$. We want to show that $T_{\bar{\gamma}_k}$ encloses the same holes as T_{γ_k}. Consider any other hole j. If j is not enclosed by T_{γ_k}, then $m_{kj} = 0$, so j is not enclosed by $T_{\bar{\gamma}_k}$ either. If j is enclosed by T_{γ_k}, then $m_{kj} = 1$, which means there is some non-boundary Dehn twist in F that encloses both k and j. But $T_{\bar{\gamma}_k}$ is the only non-boundary Dehn twist in F that encloses k, so $T_{\bar{\gamma}_k}$ also encloses j. Thus $T_{\bar{\gamma}_k}$ encloses the same holes as T_{γ_k}. By checking m_k, we can show that there is no boundary Dehn twist in F that encloses k, which implies that $T_{\bar{\gamma}_k}$ satisfies the desired properties, and we are done. ■

Before presenting our proof of the theorem about stabilizing D_n, we remind the reader of the stabilization process. We will begin with D and $\phi_0 = \mathrm{id}$ as the boundary monodromy. We only allow stabilizations that do not form any genus. After stabilizing once, our surface is extended to T_1, and the monodromy becomes $\phi_1 = T_{\gamma_1}$, where T_{γ_1} encloses hole 1. After stabilizing n times, we obtain the surface D_n, with monodromy $\phi_n = T_{\gamma_1} T_{\gamma_2} ... T_{\gamma_n}$ (see Figure **??**).

Theorem 4.7. *Let $\phi_n \in \mathrm{MCG}(D_n)$ be a product of Dehn twists that arise from stabilizing D n-times, so $\phi_n = T_{\gamma_1} T_{\gamma_2} ... T_{\gamma_n}$. Say $\overline{\phi_n}$ is a factorization of ϕ_n into positive Dehn twists. Then $\overline{\phi_n}$ consists of the n Dehn twists $T_{\bar{\gamma}_1}$, $T_{\bar{\gamma}_2}$, ..., $T_{\bar{\gamma}_n}$, where $T_{\bar{\gamma}_i}$ encloses the same holes as T_{γ_i}.*

Proof. We will prove the theorem by induction. Begin with base case $n = 0$. Then our surface is D, and the monodromy is $\phi_0 = \mathrm{id}$. Because $\mathrm{MCG}(D)$ is trivial, the theorem is obviously true.

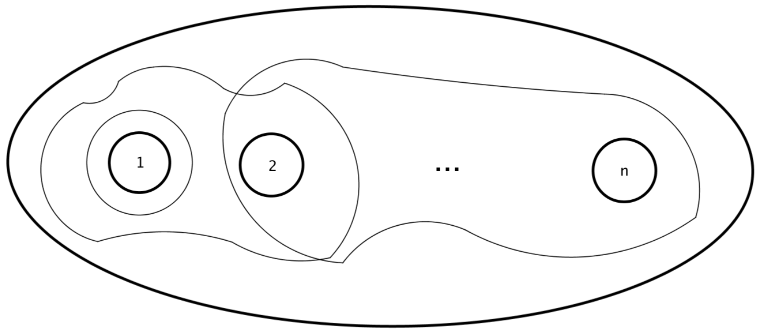

Figure 8: The figure shows D_n with ϕ_n formed by stabilizing (D,id) n times, as well as γ_1, γ_2 and γ_n. An important point to bear in mind is that γ_n is the only curve that encloses hole n.

Then we consider the inductive case, so we assume the theorem is true from $n \leq k-1$. Let us consider the surface T_{k-1} with monodromy $\phi_{k-1} = T_{\gamma_1}T_{\gamma_2}...T_{\gamma_{k-1}}$. We stabilize the surface, forming T_k with monodromy $\phi_k = T_{\gamma_1}T_{\gamma_2}...T_{\gamma_{k-1}}T_{\gamma_k}$. Now notice, by the nature of the stabilization process, that T_{γ_k} is the only Dehn twist among all the T_{γ_i} that encloses the hole k. Thus, by Lemma 4.6, we know that if we have a factorization $\overline{\phi_k}$ of ϕ_k in positive Dehn twists, then $\overline{\phi_k} = \overline{\phi_{k-1}}T_{\bar{\gamma}_k}$, where $T_{\bar{\gamma}_k}$ is the only Dehn twist in $\overline{\phi_k}$ that encloses hole k, and $T_{\bar{\gamma}_k}$ encloses the same holes as T_{γ_k}. As a result, $T_{\bar{\gamma}_k}$ has the same set of $\{m_{ij}, m_i\}$ as T_{γ_k}, and $\overline{\phi_{k-1}}$ has the same set of $\{m_{ij}, m_i\}$ as ϕ_{k-1}, so $\overline{\phi_{k-1}}$ must be a factorization of ϕ_{k-1}. By our induction hypothesis, $\overline{\phi_{k-1}}$ contains the $k-1$ Dehn twists $T_{\bar{\gamma}_1}$, $T_{\bar{\gamma}_2}$, ..., $T_{\bar{\gamma}_{k-1}}$. Because $\overline{\phi_k} = \overline{\phi_{k-1}}T_{\bar{\gamma}_k}$, $\overline{\phi_k}$ contains the k Dehn twists $T_{\bar{\gamma}_1}$, $T_{\bar{\gamma}_2}$, ..., $T_{\bar{\gamma}_k}$ as desired, and our proof is complete. ∎

To summarize, the product of Dehn twists formed by stabilization have to be factorized into the same number of Dehn twists, because the product satisfies no relation which changes the length of its factorization. For D_n, we were able to formalize this idea with the invariant given by Plamenevskaya and Van Horn-Morris (cite). Moreover, we worked in $Ab(\mathrm{MCG}(D_n))$, which motivated us to consider abelianizations of other mapping class groups and their connections with the general stabilization property.

The idea of Theorem 4.7 is also useful in proving theorem 3.6 in the case of D_n. Here we want to begin with D_n with monodromy some arbitrary $\bar{\phi} \in \mathrm{MCG}(D_n)$. We have:

Theorem 4.8. *Let $\phi_{n+m+1} \in \mathrm{MCG}(D_{n+m+1})$ be a product of Dehn twists that arise from stabilizing D_n with monodromy being an arbitrary $\phi_n \in \mathrm{MCG}(D_n)$ with one negative stabilization T_δ^{-1} and m positive stabilizations, so $\phi_{n+m+1} = \phi_n T_\delta^{-1} T_{\gamma_1}...T_{\gamma_m}$. Then ϕ_{n+m+1} cannot be factorized into a product of positive Dehn twists.*

Proof. We will prove the theorem by contradiction, so we assume there exists a $\overline{\phi_{n+m+1}}$, which is a factorization of ϕ_{n+m+1} into positive Dehn twists. We know that $\phi_{n+m+1} = \phi_{n+1}T_{\gamma_i+1}...T_{\gamma_m}$, where $\phi_{n+1} = \phi_n T_\delta^{-1}$. Thus $\overline{\phi_{n+m+1}}$ must consist of $\overline{\phi_{n+1}}$, $T_{\bar{\gamma}_1}$, ..., $T_{\bar{\gamma}_m}$, where $\overline{\phi_{n+1}}$ is a factorization of ϕ_{n+1} into positive Dehn twists, and $T_{\bar{\gamma}_j}$ encloses the same holes as T_{γ_j} (the argument is essentially the same as the inductive argument used to prove theorem 4.7: apply lemma 4.6 repeatedly). Because $\overline{\phi_{n+1}}$ consists only of positive Dehn twists, all m_{ji} of $\overline{\phi_{n+1}}$ must be non-negative. However, because T_δ^{-1} is the only (negative) Dehn twist in ϕ_i that encloses some hole i, we know that $m_{ji} = -1$, for any other hole j enclosed by T_δ^{-1}. Thus we have a contradiction and the proof is complete. ■

5 Abelianization of the mapping class group

Now we consider a surface $\Sigma_{g,n}$ of genus $g \geq 1$ with $n \geq 0$ boundary components, and ask for the abelianization of its mapping class group $\mathcal{M}_{g,n}$. As we saw in the last section, this approach was fruitful for $g = 0$, and we hope that we can obtain similar results for general surfaces with the same method. However, this hope is not realized:

Theorem 5.1. *We have*

$$\mathcal{M}_{g,n}^{\mathrm{ab}} = \begin{cases} \mathbb{Z}/12\mathbb{Z} & g = 1, n = 0 \\ \mathbb{Z}^n & g = 1, n > 0 \\ \mathbb{Z}/10\mathbb{Z} & g = 2 \\ 0 & g \geq 3. \end{cases}$$

Proof. We consider the structure of the mapping class group, in particular the presentation given by Gervais [Ger].

The surface $\Sigma_{g,n}$ can be constructed by taking a torus with $2(g-1)+n$ holes S_i and attaching $g-1$ handles H_k connecting the first $g-1$ pairs of holes.

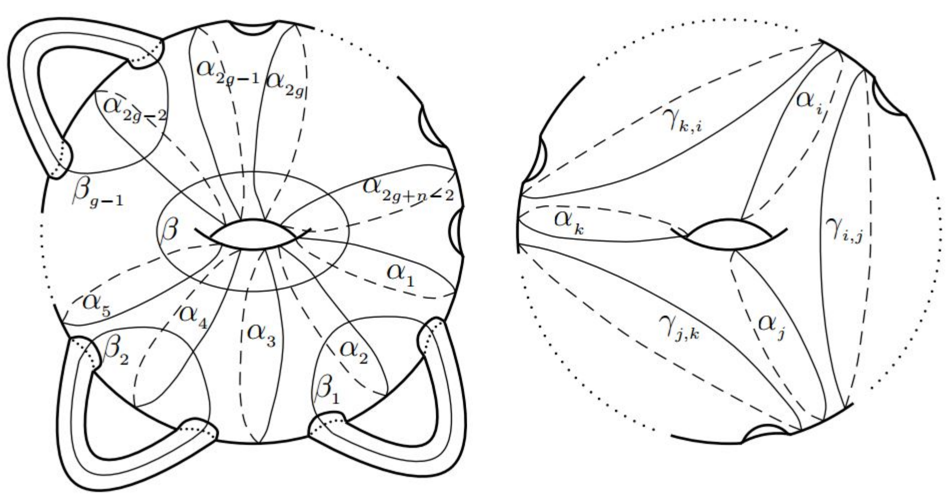

Figure 9: Curves on a surface $\Sigma_{g,n}$, constructed as in [Ger] (source for picture).

According to Gervais, $\mathcal{M}_{g,n}$ is generated by the following elements:

- the twist b around the torus longitude β;
- the twists a_i around the torus latitudes α_i between consecutive holes S_i and S_{i+1};

- the twists b_k around longitudinal curves β_k through the attached handles H_k;
- the twists c_{ij} around curves γ_{ij} on the torus that enclose the holes between α_i and α_j.

The relations are:

- handle relations $c_{(2i-1)2i} = c_{2i(2i+1)}$;
- disjointness relations $xy = yx$ for x, y generators around disjoint curves;
- braid relations $xyx = yxy$ for x, y generators around curves that intersect once;
- star relations $c_{ij}c_{jk}c_{ki} = (a_i a_j a_k b)^3$ for $i \leq j \leq k$ not all equal.

Now we pass to the abelianization $\mathcal{M}_{g,n}^{\text{ab}}$, retaining the above sets of generators and relations, but making some simplifications using commutativity and casework.

In the abelianization, the disjointness relations become trivial, and the braid relations reduce to a certain set of equalities: $a_i = b$, $b_k = a_{2k} = b$, and $c_{ij} = b_l = b$ if there exists $l \in \{i, j\}$ with $l \in \{2, \ldots, 2(g-1)\}$, i.e. α_l goes through a handle. This eliminates the generators a_i and b_k, and further restrains the c_{ij} (subsuming the handle relation, in fact).

The star relation then becomes

$$c_{ij} + c_{jk} + c_{ki} = 3(a_i + a_j + a_k + b) = 12b.$$

In the case $g = 1$, $n = 0$, this last relation reduces to $12b = 0$ and we conclude $\mathcal{M}_{1,0}^{\text{ab}} = \mathbb{Z}/12\mathbb{Z}$.

For all other pairs (g, n), we can split this class of relations into two subsets. The first, where $j \in \{i, k\}$, gives

$$c_{ki} = 12b - c_{ik}$$

for all $i < k$. The second gives $c_{ij} + c_{jk} = c_{ik}$ for all $i < j < k$, which reduces to

$$c_{i(i+1)} + \ldots + c_{(k-1)k} = c_{ik} \tag{$\star$}$$

or all $i < k$. This reduces the generating set to $b, c_{i(i+1)}$, with relations

$$c_{ik} = c_{i(i+1)} + \ldots + c_{(k-1)k} = b$$

and

$$c_{ki} = 12b - (c_{i(i+1)} + \ldots + c_{(k-1)k}) = b$$

for $i < k$, one of which lies in $\{2, \ldots, 2(g-1)\}$.

If $g = 1$, all of these relations are vacuous, so we conclude $\mathcal{M}_{1,n}^{\text{ab}} = \mathbb{Z}^n$ for $n > 1$.

If $g \geq 2$, the second relation reduces to $10b = 0$ via the first relation. If $g = 2$, then the first relation gives $c_{12} = b$ and $c_{23} + \ldots + c_{(k-1)k} = b$ for all $k > 2$, which reduces to $c_{23} = b$ and $c_{(k-1)k} = 0$ for all $k > 3$. Thus $\mathcal{M}_{2,n}^{\text{ab}} = \mathbb{Z}/10\mathbb{Z}$.

For $g \geq 3$, the first relation again gives $c_{(i-1)i} = b$ directly for all $2 \leq i \leq 2g - 1$, while $c_{(2g-2)(2g-1)} + \ldots + c_{(k-1)k} = b$ for all $k > 2g - 2$ gives $c_{(k-1)k} = 0$ for all $k > 2g - 1$, reducing the generating set to b. In this case, we have additional relations $2b = c_{23} + c_{34} = b$ giving $b = 0$. We conclude $\mathcal{M}_{g,n}^{\text{ab}} = 0$ for $g \geq 3$. This completes the proof. ■

However, there is one case where we do obtain some information from Theorem 5.1, namely the case $g = 1$, where we can prove the following proposition (a special case of Theorem 3.2).

Proposition 5.2. *There is no factorization of the identity into positive Dehn twists on the torus with $n > 0$ boundary components.*

First, we state a few preliminary lemmas. Let $\beta \subset S$ be the nonseparating simple closed curve about the latitude of the torus, and let $\alpha \subset S$ be any nonseparating simple closed curve. Cutting along α gives an annulus with n boundary components, and similarly for β. Thus, using the change-of-coordinates principle from [FM], we have:

Lemma 5.3. *There exists a homeomorphism $\phi \in \mathrm{Homeo}(S, \partial S)$ with $\phi(\alpha) = \beta$.*

This lemma implies that such T_α and T_β are conjugate in the MCG via ϕ, so that their images are equal in the abelianization.

Now suppose α is a separating simple closed curve. Cutting along α gives a decomposition $S \backslash \alpha = S_1 \sqcup S_2$. Let g_i be the genus for S_i and let n_i be the number of boundaries/punctures on $S_i \implies n_1 + n_2 = n + 2$. Note that $\chi(S) = \chi(S_1) + \chi(S_2)$ gives $-n = (2 - 2g_1 - n_1) + (2 - 2g_2 - n_2) = 4 - 2(g_1 + g_2) - n - 2$, so $g_1 + g_2 = 1$. Thus, without loss of generality, we may assume that $g_1 = 0$ and $g_2 = 1$.

Let γ be any other separating simple closed curve with decomposition $S \backslash \gamma = S_1' \sqcup S_2'$ such that S_1' contains the same boundary components as S_1. Then, again using the change-of-coordinates principle, we have:

Lemma 5.4. *There exists a homeomorphism $\phi \in \mathrm{Homeo}(S, \partial S)$ with $\phi(\alpha) = \gamma$.*

For any $A \subset \{1, ..., n\}$ let γ_A be a choice of simple closed curve enclosing boundary components C_i for $i \in A$. We have thus shown that if a nonseparating curve α encloses boundary components C_i for $i \in A \subset \{1, ..., n\}$ then $\overline{T_\alpha} = \overline{T_{\gamma_A}}$. Using equation $(\star)$ in the proof of Theorem 5.1, we obtain:

Lemma 5.5. *In the abelianization of the mapping class group,*

$$\overline{T_{\gamma_A}} = \sum_{i \in A} \overline{T_{\gamma_i}}.$$

With these lemmas, we are now ready to prove the proposition.

Proof. Note that $\mathrm{MCG}^{\mathrm{ab}}(S) = \langle b, c_i | 12b = \sum_{i \leq n} c_i \rangle = \mathbb{Z}^n$. Suppose that $\mathrm{id} = \prod T_\gamma$; then, passing to $\mathrm{MCG}^{\mathrm{ab}}$, we obtain

$$\sum \overline{T_\gamma} = m_0 b + \sum_{i \leq n} m_i c_i = 0,$$

where the m_i are some nonnegative integers. But $c_n = 12b - \sum_{i<n} c_i$, so we obtain

$$(m_0 + 12m_n)b + \sum_{i<n} (m_i - m_n)c_i = 0.$$

Since $\{b, c_i | i < n\}$ form a basis for the abelian group $\mathrm{MCG}^{\mathrm{ab}}$, we conclude $m_0 + 12m_n = 0$ and $m_i = m_n$. But m_0 and m_n are nonnegative, so we must have $m_i = 0$ for all i, and the original product of Dehn twists must have been trivial, finishing the proof. ■

6 Hyperbolic proof of symplectic results

Throughout we will use Σ to denote a surface with $\partial\Sigma \neq \varnothing$. We will say that an arc $\alpha \subset \Sigma$ is *essential* if it is not isotopic (rel endpoints) to an arc $\alpha' \subset \partial\Sigma$, i.e. if it is not isotopic to an arc contained in a boundary component.

6.1 Preliminary results

Many of the notations, definitions, and claims in this section are taken directly from [Hon].

Let $\alpha, \beta \subset \Sigma$ be essential arcs with $\alpha(0) = \beta(0) = x_0 \in \partial\Sigma$. Consider lifts $\tilde{\alpha}, \tilde{\beta}$ of α, β respectively in the universal cover $p : \tilde{\Sigma} \twoheadrightarrow \Sigma$ with $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}_0$. Note that the interior of $\tilde{\Sigma}$ is homeomorphic to the interior of a disk and $\tilde{\alpha}$ divides it into left and right regions, determined by the orientation of α. See Figure 11b for an example – the vertical arc corresponds to a lift that divides the universal cover into two regions.

Definition 6.1. *Given essential arcs $\alpha, \beta \subset \Sigma$ with $\alpha(0) = \beta(0) = x_0 \in \partial\Sigma$ and lifts $\tilde{\alpha}, \tilde{\beta} \subset \mathbb{H}$ satisfying $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}_0$, we say β* lies to the right of α *if $\tilde{\beta}(1)$ lies in the region to the right of $\tilde{\alpha}$. We will denote this as $\beta \geq \alpha$. We say β* lies strictly to the right of α *if $\tilde{\beta}(1)$ lies in the region to the right of $\tilde{\alpha}$ and does not coincide with $\tilde{\alpha}(1)$. We will denote this by $\beta > \alpha$.*

Proposition 6.2. *The previous definition is well-defined up to isotopies fixing $\partial\Sigma$.*

Proof. Suppose an essential arc γ' is isotopic to γ relative to $\partial\Sigma$. Then $\gamma'(1) = \gamma(1) = z \implies \gamma, \gamma'$ differ by an element in $\pi_1(\Sigma, z)$. However, they are homotopic, so we must have $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$. Since endpoints of lifts of arcs in the cover do not change under isotopies in Σ, we conclude that the definition is well-defined. ■

Definition 6.3. *A diffeomorphism $f \in Diffeo^+(\Sigma, \partial\Sigma)$ is* right veering *if $f(\alpha) \geq \alpha$ for every essential arc $\alpha \subset \Sigma$. We denote this subset $Veer(\Sigma)$.*

Claim 6.4. *$Veer(\Sigma)$ is a monoid under composition.*

Let $Dehn^+(\Sigma)$ denote the monoid generated by positive Dehn twists in $MCG(\Sigma)$. We have the following lemma.

Lemma 6.5 ([Hon]). *$Dehn^+(\Sigma, \partial\Sigma) \subset Veer(\Sigma, \partial\Sigma)$*

We remark that elementary proofs of these results can be found in [Hon].

Suppose $\chi(\Sigma) \leq 0$. We endow Σ with a Riemannian metric such that the boundary components are geodesics. Let $\phi \in \text{Diffeo}^+(\Sigma, \partial\Sigma)$ and let $C \subset \partial\Sigma$ be a boundary component containing a point x_0. Let $\tilde{x}_0 \in \tilde{C}$ be a lift of $x_0 \in C$ in the metric universal cover. As a corollary of the pathlifting property we get a lift diffeomorphism $\tilde{\phi} \in \text{Aut}(\tilde{\Sigma})$ such that $\tilde{\phi}(z) = z$ for every $z \in \tilde{C}$. When $\phi = T_\gamma$ is a Dehn twist about some simple closed geodesic $\gamma \subset \Sigma$, we can visualize the lift $\tilde{T}_\gamma$ as follows: choose a geodesic path α and a lift $\tilde{\alpha}$ based at $\tilde{x}_0$, follow $\tilde{\alpha}$ until the first intersection with a lift $\tilde{\gamma}$, turn right and follow $\tilde{\gamma}$ until the first intersection with a lift $\tilde{\alpha}'$, make a left and repeat until we reach the end of α. This process is illustrated as the pink curve in Figure 10 for the annulus and in Figure 11b for Σ with $\chi(\Sigma) < 0$.

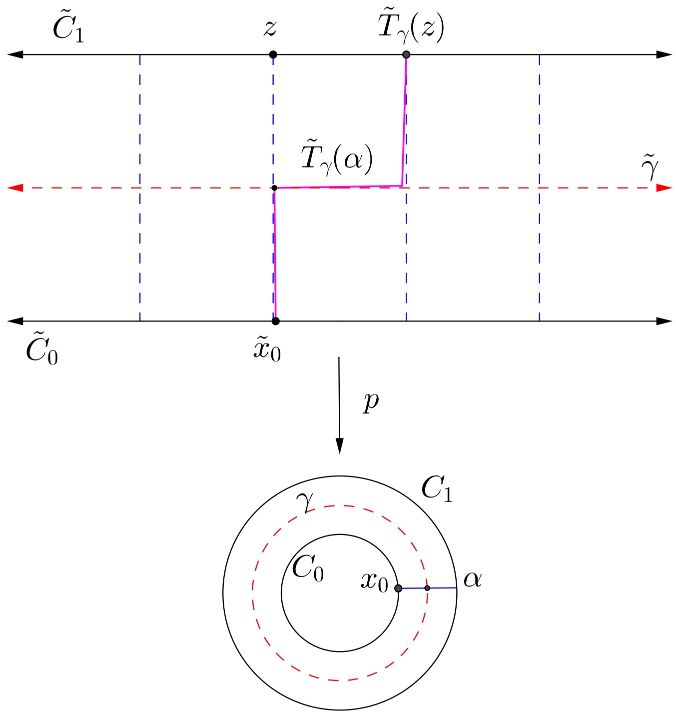

Figure 10: A lift of a Dehn twist in the annulus.

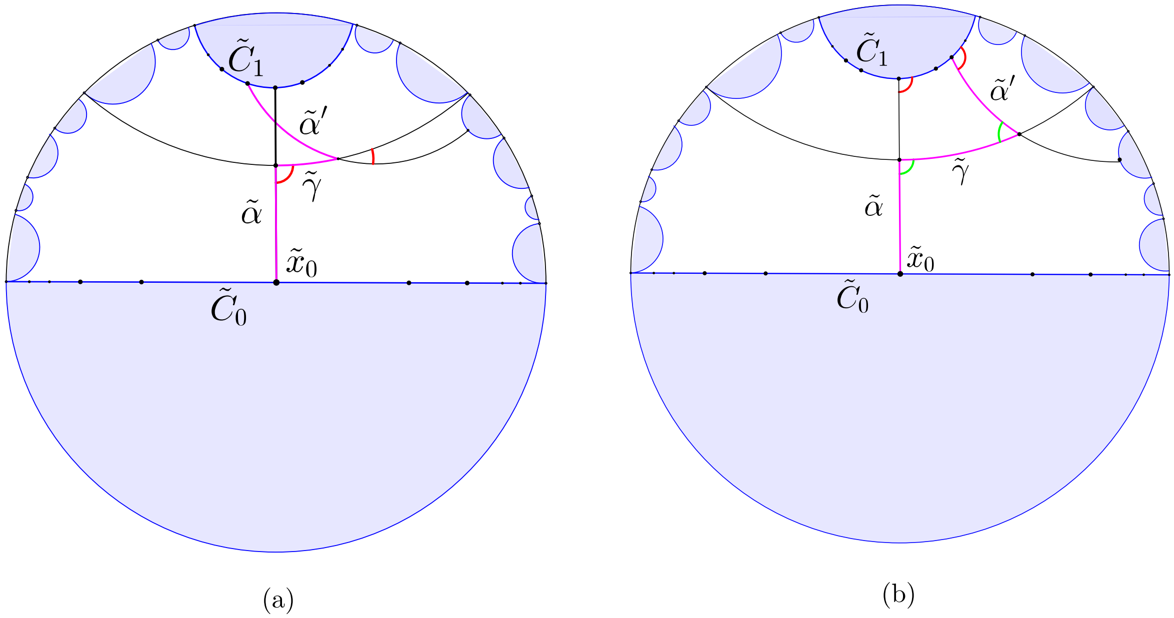

Figure 11: Proof of Lemma 6.6

We remark that when $\chi(\Sigma) < 0$, the universal cover is a subset $\tilde{\Sigma} \subset \mathbb{H}$ of the Poincaré disc $p : \tilde{\Sigma} \to \Sigma$ as shown in Figure 11b. Let $\mathcal{L} = p^{-1}(\partial\Sigma)$, which is a union of disjoint open intervals. Given an essential arc $\alpha \subset \Sigma$ with $\alpha(0) = x_0 \in \partial\Sigma$, let $\tilde{\alpha}$ be a lift based at $\tilde{x}_0$. We remark that $\tilde{\alpha}(0)$ and $\tilde{\alpha}(1)$ lie on different components of $\mathcal{L}$ since otherwise α would be homotopic to a non-essential arc. We will use the notation L_α to denote the connected component that contains $\tilde{\alpha}(1)$.

The main tool we will use is the following lemma, whose ideas were adapted from [Sm].

Lemma 6.6. *Let Σ be a surface with $\chi(\Sigma) < 0$. Let $\alpha \subset \Sigma$ be an essential geodesic arc based at $x_0 \in \partial\Sigma$ with lift $\tilde{\alpha}$ based at $\tilde{x}_0$, and let $\gamma \subset \Sigma$ be a simple closed curve with $i(\gamma, \alpha) \neq 0$. Then $T_\gamma(\alpha) > \alpha$. Furthermore, the lift $\tilde{T}_\gamma$ maps the connected component L_α to a connected component on the right of L_α.*

Proof. We first show that $T_\gamma(\alpha) > \alpha$. Let C_0, C_1 be the connected boundary components of $\alpha(0), \alpha(1)$ respectively. Let $\tilde{C}_0, \tilde{C}_1$ be lifts of C_0, C_1 with $\tilde{\alpha}(0) = \tilde{x}_0 \in \tilde{C}_0$ and $\tilde{\alpha}(1) \in \tilde{C}_1$.

Let β be a curve in the isotopy class of $T_\gamma(\alpha)$ obtained by performing surgery α at each intersection with γ. By Proposition 6.2, all lifts of curves in the isotopy class of $T_\gamma(\alpha)$ based at $\tilde{x}_0$ share the endpoint $\tilde{\beta}(1)$, so it suffices to consider this representative.

The lift $\tilde{\beta}$ then corresponds to the path that follows $\tilde{\alpha}$ until intersecting a lift $\tilde{\gamma}$, following $\tilde{\gamma}$ into the right region, turning left at the next intersection with a lift of α, and continuing in this fashion until the path ends. More explicitly, the intersection points in $\gamma \cap \alpha$ partition α into finitely many segments $\{\alpha_i\}_{i=1}^{n+1}$, where $n = i(\alpha, \gamma)$. Then we can write $\tilde{\beta} = \hat{\alpha}_1 * \hat{\gamma}_1 * \cdots \hat{\gamma}_n * \hat{\alpha}_{n+1}$, where $\hat{\alpha}_i$ corresponds to the lift of α_i along the zig-zag, and $\hat{\gamma}_i$ are lifts of a single loop around γ_i corresponding to the right turns in the zig zag. This process is illustrated as the pink curve in Figure 10 and Figure 11.

Let $\tilde{\alpha}_i$ be the complete geodesic arc associated to $\tilde{\alpha}_i$, and let R_i denote the region to the right of $\tilde{\alpha}_i$. We claim that $\tilde{\alpha}_{i+1} \subset R_i$ for every i. First note that $\hat{\gamma}_i \subset R_i \implies \hat{\alpha}_{i+1}(0) = \tilde{\alpha}_{i+1} \cap \tilde{\gamma}_i \in R_i$. Suppose $i(\tilde{\alpha}_i, \tilde{\alpha}_{i+1}) \neq 0$ as in Figure 11a. The surgery procedure on geodesics ensures that the angles indicated in the figure are congruent. Thus the geodesic triangle formed by $\tilde{\alpha}_i, \tilde{\alpha}_{i+1}, \tilde{\gamma}_i$ has angle sum greater than π which is a contradiction.

Next, observe that $\tilde{\alpha}_i$ and $\tilde{\alpha}_{i+1}$ cannot share an endpoint on $\tilde{C}_1$ because this would violate both the local homemorphism property of covering spaces and the angle deficit property of hyperbolic triangles.

Thus, by induction we have that $\tilde{\alpha}_{n+1} \subset R_1$, and specifically $\tilde{\beta}(1)$ lies strictly to the right of $\tilde{\alpha}(1) \implies T_\gamma(\alpha) > \alpha$.

We now show that all of $\tilde{C}_1$ gets mapped strictly to the right. Let $z \in \tilde{C}_1$ and let $\tilde{\alpha}$ be a geodesic arc from $\tilde{x}_0$ to z. Define $\tilde{\beta}$ as above. Suppose $\tilde{\beta}(1) \in \tilde{C}_1$. Then, because the angles indicated in Figure 11b are equal, summing the complementary angles in the geodesic quadrilateral formed by $\tilde{\gamma}_n$, $\tilde{\alpha}_n$, $\tilde{\alpha}n+1$, and $\tilde{C}_1$ yields an angle sum of 2π which is a contradiction. Using what we proved above, we see that $\tilde{T}_\gamma(z)$ must get mapped to a connected component strictly to the right of $\tilde{C}_1$. Because $\tilde{T}_\gamma(z)$ is continuous, we see that all of $\tilde{C}_1$ gets mapped to the same connected component to the right. ■

6.2 No factorization of the identity

In this section we will prove the following theorem using hyperbolic methods.

Theorem 6.7. *There is no nontrivial factorization of $id \in MCG(\Sigma)$ into positive Dehn twists.*

Proof. The cases $\Sigma = \mathbb{D}$ and $\Sigma = \mathcal{A}$ (i.e. when $\chi(\Sigma) \geq 0$) were covered in the planar section. This leaves the case when $\chi(\Sigma) < 0$. Suppose for contradiction that there exists a positive factorization $id = \prod_{i=1}^n T_{\gamma_i}$. Choose an essential arc α such that $i(\alpha, \gamma_n) \neq 0$. Then by Lemma 6.6 we have that $T_{\gamma_n}(\alpha) > \alpha$, i.e. the twist about γ_n moves α strictly to the right. By Lemma 6.5 we see that $id = \prod_{i=1}^n T_{\gamma_i}$ moves α strictly to the right, which is a contradiction. ■

6.3 Factorizations of handle extensions

We first prove a stronger version of the theorem covered in Section 3 that there are no factorizations of a negatively stabilized diffeomorphism into positive Dehn twists.

Theorem 6.8. *Let $\phi \in MCG(\Sigma)$ and let $\bar{\phi} = \phi \circ T_\gamma^{-1} \in MCG(\bar{\Sigma})$ be a negative stabilization. Then $\bar{\phi}$ is not right-veering. In particular, $\bar{\phi} \notin Dehn^+(\bar{\Sigma})$.*

Proof. First suppose $\Sigma = \mathbb{D} \implies MCG(\Sigma) = id \implies \phi = id$. Then negative stabilization yields $\bar{\phi} = T_\gamma^{-1}$, where γ is the simple closed curve about the core of the annulus. This is clearly not right-veering (see Figure 10).

Now let $\chi(\Sigma) \leq 0$. Suppose for contradiction that $\bar{\phi} = \phi \circ T_\gamma^{-1} \in Veer(\bar{\Sigma})$. Let c be the cocore of the handle that was attached via stabilization (see Figure 12). Observe that c is an essential arc in $\bar{\Sigma}$ and that $i(c, \gamma) \neq 0$. By Lemma 6.6 we see that $T_\gamma(c) > c \implies \phi(c) = \bar{\phi} \circ T_\gamma(c) > c$ by Lemma 6.5. However, $\phi \in MCG(\Sigma) \implies \phi(c) = c$, which is a contradiction. ■

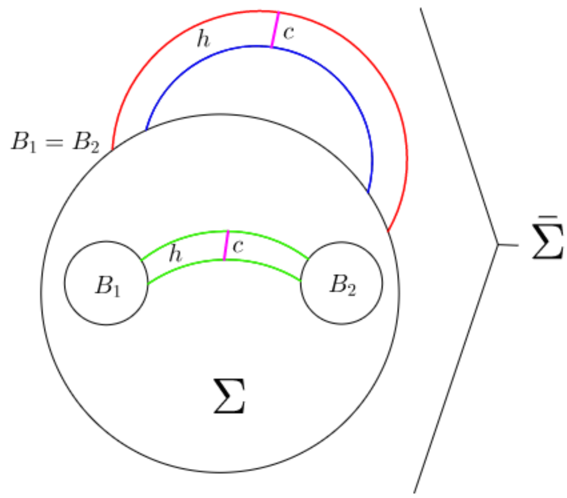

Figure 12: Attaching 1-handles to Σ

Now suppose that instead of negatively stabilizing, we add a 1-handle h with cocore c and extend ϕ by the identity, i.e. $\bar{\phi} = \phi \cup id_h$ (see Figure 12).

Theorem 6.9. *If* $\bar{\phi} = \prod_{i=1}^{n} T_{\gamma_i}$ *then* $i(\gamma_i, c) = 0$.

Proof. Again, we note that c is essential. Suppose there exists a γ_i such that $i(c, \gamma_i) \neq 0$. Let T_{γ_k} be the first Dehn twist (in order of composition) in the product that satisfies $i(c, \gamma_k) \neq 0$. By Lemma 6.6 we see that $T_{\gamma_k}(c) > c \implies \bar{\phi}(c) > c$. However, by definition we had $\bar{\phi}$ acting trivially on c since it was extended by the identity which is a contradiction. ■

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