List of Definitions, Concepts, and Formulas for Math 125

Definitions and Concepts

Chapter 13

Probability:

The chance of something gives the percentage of time it is expected to happen, when the basic process is done over and over again, independently and under the same conditions. It's the chance in the long run. Think of the probability as the frequency in infinitely many trials.

Chances are beween 0% and 100%. If something is impossible, its chance is zero percent; if something is sure to happen its chance is one hundred percent.

Probabilities may be expressed as a percent, a fraction, or a deciomal.

For example: the chance that a heart is drawn from a well-shuffled deck may be given as 25%, which is the same chance as 1/4 or 0.25.

When you draw at random from a box with tickets, all the tickets in the box have the same chance to be picked. This is called a *box model.*

For example, the chance of drawing the letter Q from a box containing one ticket for each letter of the alphabet is exactly 1/26.

The Chance of the Opposite Thing: To find the chance of 'not' something (its opposite), subtract the thing's percentage chance from 100%.

For example: The chance of getting exactly one head when three fair coins are tossed is 37.5%. So the chance of not getting exactly one head when three fair coins are tossed must be

$100\% - 37.5\% = 62.5\%$.

Conditional Probability; The chance of an event, given some other event.

It is written: P(B|A).

For example: two tickets are drawn from a box which initially had one ticket for each letter of the alphabet and placed on the table on top of each other.

Find the chance that the second ticket drawn (the top ticket) is the W, given that the first ticket drawn (the bottom ticket) was the Q. This may be written as P(W|Q). Answer: 1/25.

Unconditional Probability: A chance that puts no conditions on some other event, written P(A).

For example: two tickets are drawn from a box which intially had one ticket for each letter of the alphabet and placed on the table on top of each other.

Find the chance that the second ticket drawn (the top ticket) will be the W. No conditions on the first (bottom) ticket! This chance may be written as P(W). Answer: 1/26

Since nothing is assumed about the first ticket, we have to say that the second ticket is equally likely to be any one of the 26 letters of the alphabet.

For additionl convincing, draw the two tickets and then switch them. The ticket now on top has 1/26 chance to be the W, since the same randomization is there whether we put them in original order or switched order. All letters are equally likely to be in either position.

For additional examples, please refer to Example 2 and Exercise B1 on pages 226 and 227.

The Logic of Conjunction (and):

'A and B' can be stated as both A and B.

'A and B and C' can be stated as all of A, B, and C or as every one of A, B, and C.

The Multiplication Rule: The chance that two things will both happen equals the chance that the first will happen, multiplied by the chance that the second will happen given the first has happened. P(A and B)= P(A) $\times$ P(B|A).

For example: A deck is shuffled and two cards are dealt and placed down together on the table. Find the chance that both cards are hearts.

Answer: P(H and H) = P(H) $\times$ P(H|H) = $13/52 \times 12/51 = 1/17 \approx 6\%$.

Independent Events: Two things are independent if the chances for the second given the first are the same, no matter how the first one turns out. Otherwise, the two things are dependent.

For example: A die is rolled twice. Are the rolls independent?

Answer: Yes. The dice have no memory.

For example: Two cards are drawn with replacement from a well-shuffled deck. Are the draws independent?

Answer: Yes, it does not matter what was drawn first; the chances for the second card will be the same.

For independent events:

- It does not matter how the first thing turned out; the chances for the second event are always the same.
- P(B|A) = P(B).
- When calculating P(A and B), conditional chances are not needed.
- P(A and B) = P(A) $\times$ P(B). If two things are indpendent, the chance that both will happen equals the product of their unconditional probabilities. This is a special case of the multiplication rule.
- The subsequent event has 'no memory' of what happened previously.

When drawing at random with replacement, the draws are independent. Without replacement, the draws are dependent.

When finding the chance that both A and B happen, remember to multiply. Logical conjunction is not the same as addition; the probabilities are multiplied.

Chapter 14

Probability continued:

Listing the ways;

A pair of dice are thrown. What is the chance of getting a total of 10 spots?

Answer: Let one die be white and the other black. Make a list of the 36 possible equally-probable ways for the pair of dice to land. Three of them add to 10 spots: 6 on W and 4 on B; 5 on W and 5 on B; 4 on W and 6 on B. Divide the number of ways for the total to be 10 by the total number of ways. The chance is 3/36 or 1/12.

Mutually Exclusive: Two things are mutually exclusive when the occcurrence of one prevents the occurrence of the other; one excludes the other.

- One excludes the other.
- Both of them cannot happen.
- P(A and B) = 0.

Example: A card is dealt off the top of a well-shuffled deck. The card might be a picture card. Or it might be an ace. Are these two possibilities mutually exclusive? Answer: Yes, because the picture cards are only the Jack, Queen, and King.

Example: A box contains tickets numbered 1 to 10. One ticket is drawn at random from that box. Are the results the ticket is even-numbered and the ticket has a number greater than 7 mutually exclusive? Answer: No; the ticket could turn out to be both if it is 8 or 10.

The Logic of Disjunction (or):

'A or B' (meaning either A or B or both) can be stated in the following ways:

- A or B
- A or B or both
- At least one of: A, B
- Either A or B or both
- A and/or B

'A or B or C' can be stated as:

- A or B or C
- At least one of: A, B, C

The Addition Rule: To find the chance that at least one of two things will happen, check to see if they are mutually exclusive. If they are, add the chances.

If you want to find the chance that at least one event occurs, and the events are not mutually exclusive, do not add the chances: the sum will be too big.

Example: A card if dealt off the top of a well-shuffled deck. There are 3 chances out of 13 for it to be a picture card. There is 1 chance out of 13 for it to be an ace. What is the chance for it to be a face card (picture card or ace)? Answer: 4/13. (Picture card and ace are mutually exclusive.)

Example: A box contains 10 tickets, numbered 1 through 10. A single ticket is drawn at random from the box. True or false: the chance of getting either an even-numbered card or a card greater than 7 is $1/2 + 3/10 = 5/10 + 3/10 = 8/10 = 4/5$. Answer: False, the results are not mutually exclusive, so adding the chances will give the wrong answer. (Note: the actual answer is found by enumerating all the cards that are even or greater than 7: 2, 4, 6, 8, 9, 10. There are six of them out of ten tickets, so the correct answer is 6/10=3/5.)

The Opposite of a Disjunction:

The opposite of A or B or C is not A and not B and not C.

The opposite of either A or B is neither A nor B (that's not A and Not B).

The opposite of at least one of A, B, C is none of A, B, and C.

The event "every one of A, B, and C did not happen" is the same as "none of A, B, and C." So, the event every one of A, B, and C did not happen is the opposite of at least one of A, B, C.

If the chance of an event is hard to find, try to find the chance of the opposite event.

Examples of Equivalent Expressions and of Expressions of Opposite Events:

A fair die is rolled three times. Do not confuse "all 6's," "not all 6's," "some 6's," and "no 6's."

1. Compare each of (a) to (f) to "not all sixes." Decide if it is the same, the opposite, or neither.
   - (a) No sixes.
   - (b) All sixes.
   - (c) Some sixes.
   - (d) Some non-sixes.
   - (e) At least one non-six.
   - (f) A non-six on the 1st roll or a non-six on the 2nd roll or a non-six on the 3rd roll.
2. Compare each of (a) to (e) to "all non-sixes." Decide if it is the same, the opposite, or neither.
   - (a) No sixes.
   - (b) All sixes.
   - (c) Some sixes.
   - (d) At least one six.
   - (e) A six on the first roll or a six on the seocnd roll or a six on the third roll.

Answers: 1. Same: (d), (e), (f); Opposite: (b); Neither: (a), (c). 2. Same: (a); Opposite: (c), (d), (e); Neither: (b).

Example: A coin is tossed and a die is rolled. Find the chance that either the coin lands heads up or the die lands with the $\boxed{1}$ up.

Answer: the sum of $1/2 + 1/6 = 2/3$ is not the correct answer, as the results are not mutually exclusive.

One method is to instead start by finding the probability of the opposite event. The opposite event is that the coin landed tails and the die did not land with the one-spot up.

The things are independent, so just multiply the unconditional chances: $1/2 \times 5/6 = 5/12$.

Next, subtract that chance from 1 to get $1 - 5/12 = 7/12$.

Example:

Three cards are drawn at random from a deck of cards and placed up on the table. (This assumes that the draws are made without replacement.)

Find the chance that all three cards are numbered cards (2 through 10).

i. Had the draws been made with replacement would the chance that all three of them are numbered have been smaller, greater, or the same?

Answers:

Since the draws are not independent, conditional probabilities are needed.

$$\begin{aligned} \text{P(N and N and N)} &= \text{P(N)} \times \text{P(N|N)} \times \text{P(N|NN)} = (36/52) \times (35/51) \times (34/50) \\ &= (4 \times 3 \times 3)/(13 \times 4) \times (7 \times 5)/(17 \times 3) \times (17 \times 2)/(5 \times 5 \times 2) = 21/65 \approx 32.3\%. \end{aligned}$$

i. If the draws were made with replacement, the chances would have been slightly greater, because for the second and third draws there would be a greater proportion of numbered cards in the deck. (The actual chance is about 33.2%.)

Example:

Three cards are drawn at random from a deck of cards and placed up on the table. (This assumes that the draws are made without replacement.)

Find the chance of getting at least one heart among the 3 cards.

True or false: Since the chance of getting a heart on each of the three draws is 25%, the chance of at least one heart equals P(heart on 1st draw or heart on 2nd draw or heart on third draw) = P(H) + P(H) + P(H) =25% + 25% + 25% =75%.

If this is false, find the correct chance of getting at least one heart among the 3 cards .

i. Had the draws been made with replacement would the chance of at least one heart have been smaller, greater, or the same?

Answers:

Since the draws are not mutually exclusive, it is not correct to add the chances. The proposed answer of 75% is too big. Therefore, the answer is False.

One way to approach this problem is to use the method of the Chevalier de Méré.

First find the chance of the opposite thing. Here it is that none of the three draws results in a heart. Without replacement, the draws are not independent and conditional probabilities must be used in the calculation.

P(not H and not H and not H) = P(not H) $\times$ P(not H|not H) $\times$ P(not H|not H and not H) = $(39/52) \times (38/51) \times (37/50)$, which reduces to $(19 \times 37)/(2 \times 17 \times 50) = 703/1700$.

The chance of the opposite event is 1 - 703/1700 = 997/1700 $\approx$ 58.647%.

i. If the draws were made with replacement, the chances would have been slightly smaller.

To get at least one heart when the first draw did not result in a heart, one needs a heart on the second or third draw. The cards being replaced in the critical cases are not hearts, so there would be a smaller proportion of hearts in the deck for those two draws. (The actual chance is about 57.81%.)

Example.

A box has five tickets, one marked with a star, and the other four blank:

$$\boxed{\boxed{\star}\,\square\,\square\,\square\,\square}$$

Two draws are made at random without replacement from this box. The statistician wants to find the chance of getting the star at least once in the two draws.

True or false.

1. The results 'getting the star on the first draw' and 'getting the star on the second draw' are mutually exclusive.

   Answer: True.

2. If two events A and B are mutually exclusive, the probability of at least one of A and B happening equals the sum of their chances:

   $$\mathrm{P(A\ or\ B) = P(A) + P(B).}$$

   Answer: True.

3. The chance of getting the star at least once in the two draws is (choose one answer and explanation):

   A. $$1/5 + 1/5 = 20\% + 20\% = 40\%.$$

   Add the probabilities: 20% that the star is chosen on the first draw and 20% (1 out of 5) that the star is chosen on the second draw. This requires the unconditional chance that the star will be chosen on the second draw. Do not make the mistake of assuming that a blank has already been drawn; that would be a conditional chance and is not what is called for in the addition rule.

   B. $$1/5 + 1/4 = 20\% + 25\% = 45\%.$$

   Add the probabilities: 20% that the star is chosen on the first draw and 25% (1 out of 4) that the star is chosen on the second draw. The star can only be drawn second if a blank ticket was drawn first.

   C. $$1 - (4/5)(4/5) = 1 - 16/25 = 9/25 = 36\%.$$

   Find the chance of the opposite event: $4/5 \times 4/5$, which equals 16/25 or 64%. Then subtract that from 100% to get 36%.

   D. $$1 - (4/5)(3/4) - 1/3/5 = 2/5 = 40\%.$$

   Find the chance of the opposite event: getting a star on the first draw and a star on the second draw (using conditional probabilities): $4/5 \times 3/4 = 3/5 = 60\%$. Then subtract that from 100% to get 40%.

   The correct answers are (A) and (D). It is possible to arrive at the correct chance by either of two different methods.

Example (a):

The unconditional probabilities of events A, B, and C are, respectively: 2/5, 1/4, and 1/5.

A. Assuming A, B, and C are independent, find the chance that all of them happen.

B. Assuming A, B, and C are mutually exclusive, find the chance that all of them happen.

Answers to Example (a):

A. Just multiply their unconditional probabilities:
$(2/5) \times (1/4) \times (1/5) = 2/100 = 0.02 = 2\% = 1/50.$

B. Being mutually exclusive means that the chance of any two of them happening is 0.

Example (b):

The unconditional probabilities of events A, B, and C are, respectively: 2/5, 1/4, and 1/5.

1. Assuming A, B, and C are independent, find the chance that at least one of them happens.
2. Assuming A, B, and C are mutually exclusive, find the chance that none of them happens. (This is the same as the chance that each of them did not happen.)

Answers to Example (b):

1. First Observe: Since P(A and B) = P(A) $\times$ P(B) = $(2/5) \times (1/4) = 1/10$ (and not 0), A and B are not mutually exclusive. So, the correct answer cannot be $40\% + 25\% + 20\% = 85\%$.

   The opposite of at least one is that none of them happen.

   Assuming independence of the opposite events, this is:
   $$\begin{aligned}\text{P(not A and not B and not C)} &= \text{P(not A)} \times \text{P(not B)} \times \text{P(not C)} \\ &= (3/5) \times (3/4) \times 4/5 = 36/100 = 36\%.\end{aligned}$$
   The desired chance (the opposite of this) is just $100\% - 36\%$: 64%.

2. The opposite of none of them happening is at least one happening.

   Because they are mutually exclusive that chance is just $40\% + 25\% + 20\% = 85\%$.

   The chance of none of them happening is now $100\% - 85\% = 15\%$.

   (The chance that none of them happen cannot be found by multiplication, as was done in part 1., because in this part, independence cannot be assumed.)

Chapter 15

The Binomial Formula:

Example: A fair die is rolled 16 times. Which is more likely 2 $\boxed{1}$'s or 3 $\boxed{1}$'s?

Answer: For $k = 2$, the numbers are $\frac{16!}{2! \times 14!} \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^{14} = 120 \times 1/36 \times (5/6)^{14} = 0.260.$

For $k = 3$, the numbers are $\frac{16!}{3! \times 13!} \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^{13} = 560 \times 1/216 \times (5/6)^{13} = 0.2423.$

The result with 2 $\boxed{1}$'s is more likely.

Formulas

The Chance of the Opposite

The chance of something equals 100% minus the chance of the opposite thing.

P(A) = 100% − P(not A). Also, P(not A) = 100% − P(A).

In decimal or fraction form: P(A) = 1 − P(not A), or P(not A) = 1 − P(A).

The Multiplication Rule

The chance that two things will both happen equals the chance that the first will happen, multiplied by the chance that the second will happen given that the first has happened. **P(A and B) = P(A) × P(B|A)**.

Independence

Two things are *independent* if the chances for the second given the first are the same, no matter how the first one turns out. Otherwise, the two things are *dependent*.

When drawing at random with replacement, the draws are independent. Without replacement, the draws are dependent.

The Chance that Both of Two Independent Things Will Happen

Special case when A and B are independent (meaning that P(B|A) = P(B)):

$$\mathrm{P}(\text{both A and B}) = \mathrm{P}(\text{A and B}) = \mathrm{P}(\mathrm{A}) \times \mathrm{P}(\mathrm{B})$$

This may be stated verbally:

"If two events are independent, the chance that both will happen equals the product of their unconditional probablilites. This is a special case of the multiplication rule."

There are three important points here.

- Conditional probabilities are not needed when the events are independent.
- The chance being calculated is that both A *and* B will happen. The conjuction is '*and*'.
- Blindly multplying chances can make real trouble. Be sure to check for independence, or use conditional probabilities.

Mutually Exclusive: Definition

Two things are *mutually exclusive* when the occurrence of one prevents the occurrence of the other: one excludes the other.

The Addition Rule

To find the chance that at least one of two things will happen, check to see if they are mutually exclusive. If they are, add the chances.

The Chance that At Least One of Two Things Will Happen

Special case when A and B are muutally exclusive (meaning that P(A and B) = 0);

$$\text{P(at least one of A and B)} = \text{P(A or B)} = \text{P(A)} + \text{P(B)}$$

This may be stated verbally:

"If two events are mutually exclusive, the chance that either will happen equals the sum of their unconditional probablilites." This is called the "addition rule."

There are two important points here.

- When calculating the chance of B, do not assume that A has already happened: the formula does not call for the conditional probablility of B given A, but rather P(B), the unconditional chance of B happening.
- The chance being calculated is that either A *or* B will happen.

  There are several ways of stating this: 'either A or B', 'at least one of A or B', 'A or B or both', 'A and/or B', or just 'A or B'. The conjuction is always '*or*'.

The Chance that at Least One of Several Things Will Happen

Special Case: When the things are mutually exclusive.

The chance that at least one of those things will happen reduces to the chance that exactly one of them will happen—the things being mutually exclusive means that there is no chance that more than one of them will happen. This chance can be calculated by the addition rule.

The Chance of A or B, when they are Not Mutually Exclusive

If you want to find the chance that at least one event occurs, and the events are not mutually exclusive, do not add the chances: the sum will be too big.

Blindly adding the chances can give the wrong answer, by double-counting the chance that both things happen. With mutually exclusive events, there is no double-counting.

There are several methods to find the chance of either of two events when the events are not mutually exclusive. One is the method of the Chevalier de Méré, presented below as the "Procedure when the things are not mutually exclusive." When applying that procedure, it is good to know that the opposite of 'A or B' is 'not A and not B,' or—as otherwise stated—the opposite of 'either A or B' is 'neither A nor B.'

The Usefulness of the Opposite Event:

If the chance of an event is hard to find, try to find the chance of the opposite event. Then subtract from 100%. (See The Chance of the Opposite, above.) This is useful when the chance of the opposite is easier to compute.

The Chance that at Least One of Several Things Will Happen

(If you want to find the chance that at least one event occurs, and the events are not mutually exclusive, do not add the chances: the sum will be too big. Blindly adding the chances can give the wrong answer, by double-counting chances where two or more of the things happen together. With mutually exclusive events, there is no double-counting.)

Procedure when two or more things are not mutually exclusive:

- The opposite thing is that none of those things will happen. This, the opposite thing, means that each of the original things did *not* happen. Its chance can now be calculated by repeated application of the multiplication rule. (With 2 things, only 1 application is needed.)
- The original chance that we desired—namely the chance that at least one of several things will happen—can now be found by subtracting the chance of its opposite from 100%.

Summary of the de Méré procedure for "at least one":

A) It might be easier to find the chance of the opposite event, especially when the events A, B, C, etc. are not mutually exclusive.

B) The opposite of "at least one" (same as A or B or C) is "none."

C) Find the chance of "none" (same as not A and not B and not C) using the multiplication rule.

D) This will be: P(none) = P(not A) × P(not B) × P(not C).

E) Subtract that result from 1 (or 100%).

The Binomial Formula

The chance that an event will occur exactly k times out of n is given by the binomial formula

$$\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$$

In this formula, n is the number of trials, k is the number of times the event is to occur, and p is the probability that the event will occur on any particular trial. The assumptions:

- The value of n must be fixed in advance.
- p must be the same from trial to trial.
- The trials must be independent.

The first factor in the binomial formula is the binomial coefficient.