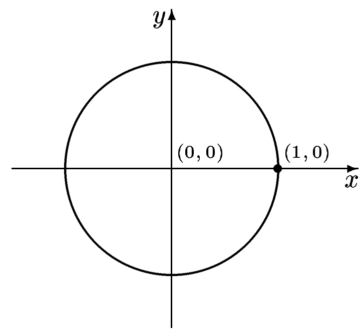
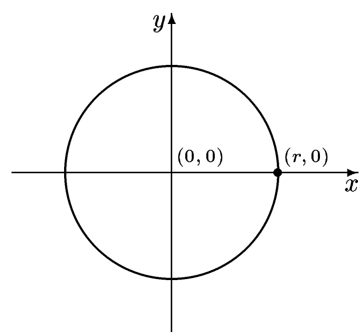
Circles

Math 130 *Kovitz* 2012

The graph of $x^2 + y^2 = 1$ is a unit circle with center $(0, 0)$ and radius 1.



The graph of $x^2 + y^2 = r^2$ is a circle with center $(0, 0)$ and radius r.



Given any equation of the form

$$x^2 + y^2 + Ax + By + C = 0,$$

it can be put in the form

$$(x - h)^2 + (y - k)^2 = s$$

by completing the square.

For example:

$$x^2 + y^2 + 8x - 12y + 43 = 0$$
$$x^2 + 8x + y^2 - 12y = -43$$
$$(x^2 + 8x + 16) + (y^2 - 12y + 36) = -43 + 16 + 36$$
$$(x + 4)^2 + (y - 6)^2 = 9$$

This is a circle with radius 3, since $r^2 = 9$ and $x^2 + y^2 = 9$ is a circle with radius 3 and center $(0, 0)$. As we will later learn, the substitutions of $x + 4$ for x and $y - 6$ for y cause shifts of 4 to the left and 6 up. The center, originally at $(0, 0)$ is shifted 4 to the left and 6 up, landing at the point $(-4, 6)$.

Be Careful. Add the completing numbers for x and y to both sides of the equation and remember that h and k in (h, k) have signs opposite to the signs separating x and y from the numbers in the completed square version of the equation. The equation in the example above could have been rewritten as $\big(x - (-4)\big)^2 + \big(y - 6\big)^2 = 9$, with $h = -4$ and $k = 6$.

General Form of the Equation of a Circle:

When s is positive, the graph of

$$(x-h)^2 + (y-k)^2 = s$$

is a circle with center (h, k) and radius $\sqrt{s}$. Two questions remain:

1) What if $(x-h)^2 + (y-k)^2 = s$ and s is negative?

No solution is possible since the left side is positive or zero and the right side is negative. There is no graph—the solution set is empty.

2) What if $(x-h)^2 + (y-k)^2 = 0$?

The sum of two expressions, each a square and therefore greater than or equal to zero, is equal to zero. The only possibility is for each expression to equal zero. The solution set is a single point, (h, k).

For example:

$$(x-3)^2 + (y+2)^2 = 0$$

The solution set is the point $(3, -2)$ only. This case is called a degenerate circle.

The case

$$Dx^2 + Dy^2 + Ax + By + C = 0$$

has, of course, a graph which is also a circle, a single point, or no points. You just divide the expression by D as the first step.

For example:

$$3x^2 + 3y^2 + 24x - 36y + 129 = 0$$

is the same graph as the example on page 1. First divide by 3 to obtain the same equation as in the first example.

The square must be completed by taking the coefficient of the x term, dividing by 2 and then squaring. The same procedure is then used for the y term. The terms involving x^2 and y^2 should have coefficients equal to 1.

The procedure was

$$\begin{array}{rcrcrcl} 8x & \longrightarrow & 8 & \longrightarrow & 4 & \longrightarrow & 16 \\ -12y & \longrightarrow & -12 & \longrightarrow & -6 & \longrightarrow & 36. \end{array}$$

Alternate Method:

From $x^2 + y^2 + Ax + By + C = 0$ we get $h = -A/2$ and $k = -B/2$. Remember to put the $-$ in front of the h and k when they are placed into the equation.

The example redone in this alternate method would be as follows.

(for x) $\quad h = -8/2 = -4$

(for y) $\quad k = -(-12)/2 = 12/2 = 6$

$$(x+4)^2 + (y-6)^2 = -43 + (4)^2 + (-6)^2$$

The completing numbers, $(4)^2$ and $(-6)^2$, were added to both sides of the equation.