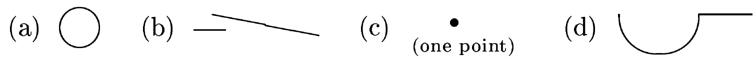
Reflections and Symmetry

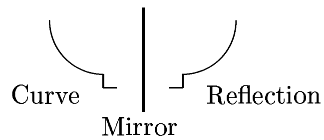
Math 130 *Kovitz*

Reflection: General Concepts

A curve is defined as a set of points. The following are therefore all examples of curves:



The process of reflection has an effect similar to that of producing an image of a two-dimensional object in a mirror.



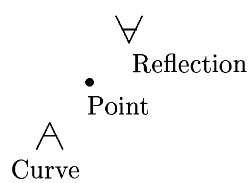
Properties of Reflections

- A reflection preserves the size and shape of the original curve. (The reflection in the mirror is the same shape—only backwards.)
- A reflection keeps some points fixed. (In the case of the mirror all points on the surface of the mirror are left fixed.)
- The reflection of the reflection is the orginal curve. (If you reflected your image back through the mirror, the result would be the original object, you.)

Given these three properties, there are only two types of reflections in the plane:

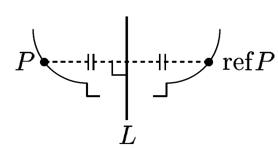
- Across a line and
- Through a point.

Here is an example of reflection through a point (an effect similar to that of focusing through a camera lens).



Procedures of Reflection

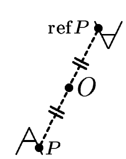
To reflect a curve across a line L, from each point P of the curve draw a line perpendicular to the line L and travel the same distance on the other side of L, obtaining ref$P$.



These points, ref$P$ for each P on the original curve, form the new, reflected curve. (This makes sense—think of a procedure to produce the image of an object in a mirror.)

Informally we flip over the paper, using L as a hinge.

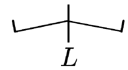
To reflect a curve through a point O, draw a line from each point P of the curve through O and continue an equal distance on the other side, obtaining ref$P$.



These points, ref$P$ for each P on the original curve, form the new, reflected curve. (This makes sense—think of a procedure to produce the image of an object in a camera lens.)

Informally we place a pin at point O and turn the paper 180°, using the pin as the point of rotation.

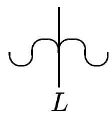
Symmetry: General Concepts

It should be intuitively clear when a figure or curve is line-symmetric. For example is symmetric because 

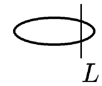
line L divides it into two parts, each the reflection of the other.

Rule 1 *A curve C is symmetric across a line L if all points of the curve not on line L form two parts, each the reflection of the other, each on one side of line L.*

Examples:



Symmetric across L

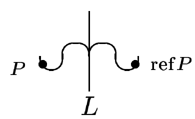


Not symmetric across L (What about across another line?)

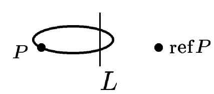
Definition 1 *A curve is symmetric across a line L if for every point P on the curve C, its reflection is also on the curve.*

This means that each point P on the curve C but not on the line L is part of a pair of "symmetric" points on the curve, that is part of a pair of points each the reflection of the other across the line L and *both on the curve C*. (These two points may be thought of as a symmetric pair relative to the line—every point on the curve is part of such a symmetric pair.)

Examples:



(True for all points P on the curve— so symmetric)



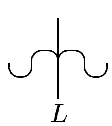
(Not on curve— so not symmetric)

- To use Definition 1 to show non-symmetry: If there is even one point on C which has its reflection not on C, then C is not symmetric.
- To use Definition 1 to show symmetry: If for every point P on C, its reflection across L is also on C, then C is symmetric across L. This is often done by taking a "typical point" of C (a point which represents all points of C) and showing that its reflection is also on C. This idea of a typical point will become clearer in reference to equations and symmetry.

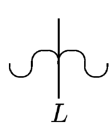
Definition 2 *A curve C is symmetric across a line L if the reflection of C is the same as C.*

This means that "flipping" the curve about the line L leaves it unchanged. It is important to note that two curves are the same if they have the same set of points. It is not enough to require that they have the same size and shape, but it is also necessary that they be in the same position. Otherwise the reflection of C would always be the same as C and the curve C would always be symmetric.

Examples:

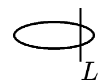


Curve C

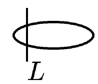


ref$C$

C and ref$C$ are the same, so symmetric.



Curve C



ref$C$

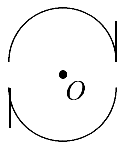
C and ref$C$ are not the same set of points, so not symmetric.

This definition will take on a new meaning when we define curves through equations. Definitions 1 and 2 could be shown to be mathematically equivalent (proof omitted). Clearly if each point on C is part of a symmetric pair of points, both on C, flipping across the line will lead to the same curve and vice versa.

Point Symmetry

It should be intuitively clear when a figure is point-symmetric.

For example:

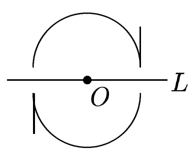


is symmetric because point O is a focusing point through which each part is a reflection of the other.

Rule 2 *A curve is symmetric through a point O if any line L through O divides all points of the curve except O into two parts with each part a point reflection of the other and each part, excepting its points which are on line L, on one side of line L.*

This is a simplified form of the rule which avoids the problem of what to do with points on line L. This rule is not very useful because it requires a line L, it is confused with the rule for line symmetry, and it requires careful definition (omitted here) of how the line divides up the points on L. I have included it for completeness and because it is essentially a restatement for point symmetry of Rule 1.

For Example:



Symmetric—each part is the reflection of the other through O.

Definitions 1 and 2 both work similarly for point symmetry. Definition 2 means that placing a pin in point O and rotating the paper 180° leaves C unchanged.