Reflections Determined from Equations

Math 130 *Kovitz*

Reflections and Coordinates

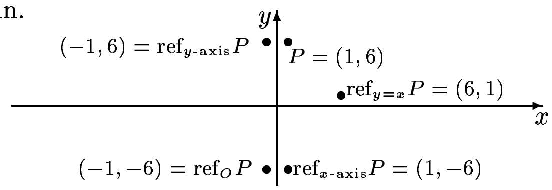
Let us consider a point in the xy-plane. There are four reflections that wil be of interest, mainly because they are easy to handle algebraically in practice. They are

- three across lines, namely across the y-axis, the x-axis, and the line $y = x$

  and

- one through a point, namely through the origin.

Take a point on the plane, such as $(1, 6)$, for example. It can be verified that the reflection of that point in the four cases will be $(1, -6)$—across the x-axis, $(-1, 6)$—across the y-axis, $(6, 1)$—across the line $y = x$, and $(-1, -6)$—through the origin.



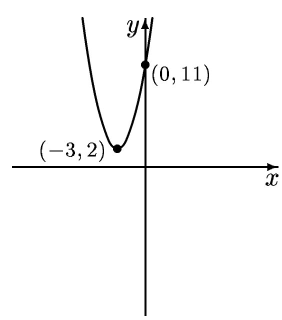
It is helpful to think of the x-axis as the equation $y = 0$. Then it makes more sense that the reflection across the x-axis changes the sign of the y-coordinate rather than the x-coordinate.

Reflections Determined from Equations

Suppose an equation is given, for example: $y = x^2 + 6x + 11$. The solution set of this equation is a relation which consists of all ordered pairs (a, b) such that substituting a for x and b for y into the equation yields a true equation. For instance $(1, 4)$ is not in the solution set, since $4 = 1^2 + 6(1) + 11$ or $4 = 18$ is not true. However $(3, 38)$ is in the solution set, since $38 = 3^2 + 6(3) + 11$ or $38 = 9 + 18 + 11$, which is the equivalent to $38 = 38$, is true.

The solution set is a set of ordered pairs. When these pairs are used as the coordinates of points, we get a set of points or curve, called the graph of the equation. We shall soon learn that this curve is:

THE ORIGINAL CURVE



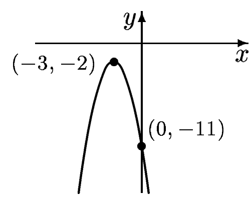
Note that $(2, 27)$ is also a point in the solution set.

Now let us consider a new equation: $-y = x^2 + 6x + 11$. I assert that since $(2, 27)$ makes the original equation true, $(2, -27)$ will surely make the new equation true. Why? Look at the results of substituting $(2, -27)$ in the new equation. We get $-(-27) = 2^2 + 6(2) + 11$ or $27 = 4 + 12 + 11$, which leads to $27 = 27$.

Since replacing each y of the original equation by $-y$ yields the new equation, replacing a point of the original curve by its reflection across the x-axis will yield a point on the new curve. This is true because the replacement of each occcurrence of y by $-y$ to produce the new equation, the replacement of b of the point (a, b) by $-b$ to produce the point $(a, -b)$, and the subsequent substitution of $(a, -b)$ into the new equation will result in exactly the same equation as the substitution of (a, b) into the original equation. The equations are exactly the same because $-(-b) = b$; wherever the original result had b, the new result will have $-(-b)$, which is also (algebraically) b. Since (a, b) was assumed to be on the original equation, the resulting equation upon substitution was true and this equation—being the same—is also true.

We can conclude that the new curve is the reflection across the x-axis of the old curve, because it consists of the reflections of all the points of the original curve (and nothing else, as could be shown).

THE REFLECTED CURVE
(across the x-axis)



Similar methods will apply for the other three reflections: y-axis, line $y = x$, and origin.

Rule 1 *Given an equation and its graph, curve C, to produce an equation of the reflection of curve C across the x-axis simply replace every occurrence of y in the original equation by $-y$.*

Symmetry Determined from Equations

When the original curve be symmetric across the x-axis? Exactly when the new, reflected curve is the same as the original curve (using Definition 2 of symmetry).

In the case of $y = x^2 + 6x + 11$ does the original curve look symmetric across the x-axis? Certainly not!

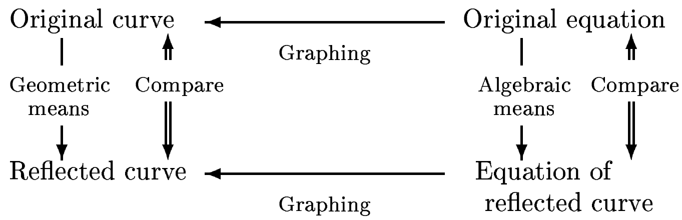
To show that it is not symmetric by geometric means there are two methods:

The first method is to show that not every point has its reflection on the curve. In this case it is obvious from the graph of the original equation that the reflection across the x-axis of the point $(-3, 2)$, which turns out to be the point $(-3, -2)$, is not on the original curve. If you wish you could also plug the point $(-3, -2)$ into the original equation $y = x^2 + 6x + 11$ to see that the resulting equation would not be true, meaning that $(-3, -2)$ is not on the original curve.

The second method is to show that the reflected curve is not geometrically the same as the original curve. (Remember that being the same means that they are both in the same location.) Here it is obvious that they are not in the same place, hence the curves are not symmetric across the x-axis.

To show symmetry or non-symmetry by purely algebraic means—from the equation only, *without graphing*—one simply finds an equation of the reflected curve and checks if it has the same graph (or set of points in the solution set) as the original curve.

Here is the relationship:



In this example the algebraic test of symmetry would go like this:
C = original curve, the graph of $y = x^2 + 6x + 11$ and
$\mathrm{ref}_x C$ = new, reflected curve, the graph of $-y = x^2 + 6x + 11$.

Do these two graphs have the same points? Obviously not, since $(2, 27)$ does not make the new equation, $-y = x^2 + 6x + 11$, true. We can conclude that the original curve is not symmetric across the x-axis—*without actually graphing.* Here Definition 2 was used: A curve C is symmetric across a line if the reflection of C is the same as C.

Similar rules apply for the other three symmetries.

Rule 2 *Given an equation and its graph, curve C, to produce an equation of the reflection of curve C across the y-axis simply replace every occurrence of x in the original equation by $-x$.*

Rule 3 *Given an equation and its graph, curve C, to produce an equation of the reflection of curve C across the line $y = x$ simply replace every occurrence of x in the original equation by y and replace every occurrence of y in the original equation by x.*

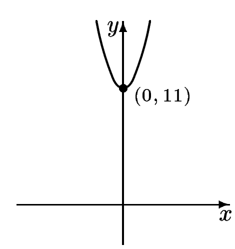
Rule 4 *Given an equation and its graph, curve C, to produce an equation of the reflection of curve C through the origin simply replace every occurrence of x in the original equation by $-x$ and replace every occurrence of y in the original equation by $-y$.*

To check for symmetry in any of these cases just compare the new equation with the original equation. If the two equations are the same or equivalent (meaning that they have the same set of points in their solution sets), then the original equation has a symmetric graph.

Usually one shows nonsymmetry by taking a point which is a solution for the original equation and showing that it is not a solution for the new equation.

Usually one shows symmetry by showing that the original equation and the new equation are algebraically the same.

As an example of an equation of a symmetric curve, take $y = x^2 + 11$ and find $\mathrm{ref}_y C$ by replacing x by $-x$ in the equation, obtaining $y = (-x)^2 + 11$ or $y = x^2 + 11$. The equation $y = (-x)^2 + 11$ when graphed gives us the reflection across the y-axis of the curve C, the graph of the original equation $y = x^2 + 11$. The algebraic equivalence of this resulting equation to the original equation means that the reflection of curve C across the y-axis is the same as curve C. That is, by definition, symmetry of curve C across the y-axis.



Graph of
$y = x^2 + 11$