Functions and Graphs

Math 130 *Kovitz*

Conventionally, let x refer to the first member of the ordered pair, the independent variable, and let y refer to the second member of the ordered pair, the dependent variable.

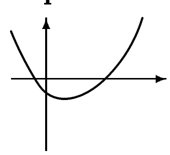
Function as a Relation: A function is a relation (a set of ordered pairs) for which all of the following three properties hold:

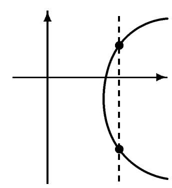
- There do not exist two ordered pairs in the relation with the same x and different values of y.

  Two ordered pairs with that property would, when graphed, be on the same vertical line. This leads to the vertical line test: No two points on the graph of a function lie on the same vertical line—that is, no vertical line meets the graph of a function at two points.

- For every x there is at most one y with (x, y) in the relation.
- For every x in the domain of the relation there is exactly one y with (x, y) in the relation.

Examples

(a)  a function, no vertical line meets it at two points

(b)  not a function, many vertical lines, including the dashed one, meet it at two points

(c) $y = x^2$: a function, given x there is exactly one y

(d) $x = y^4$: not a function, given positive x there are two values of y—such as, for $x = 16$: $(16, 2)$ and $(16, -2)$

(e) $x^2 + y^2 = 100$: not a function—if $x = 8$ is given, $(8, -6)$ and $(8, 6)$ are in the relation

Definition: If a relation is a function named f, then for every x *in the domain* the unique y is called $f(x)$. (Don't confuse $f(x)$ (f a function) with f times x.)

Be careful: The functional value $f(x)$ refers to the second member of the ordered pair when the first member is x. In other words it is the result obtained when the function f is applied to x. The idea of a function as an operator or machine is similar: f operates on an x in the domain and yields $f(x)$. The ordered pair will then be $\big(x, f(x)\big)$. No confusion will result if one refers to the operator by the term function, leaving understood that the function is actually the set of ordered pairs resulting when that operator acts on all x in the domain.

Sometimes, but not always, the relation is expressible as the solution set of an equation. When an equation of the relation has y alone on the left side and no y's on the right side, the relation is a function of x provided the expression on the right is not a compound expression like $y = \pm 3$, which contains an "or" and actually represents two equations. Thus $y = 3x^8 + 4x^5 - 16$ is a function because for a given x there can be only one y which makes the equation true, namely the result of substituting the given value of x into the right side. When $x = 2$, $y = 3 \cdot 2^8 + 4 \cdot 2^5 - 16 = 3 \cdot 256 + 4 \cdot 32 - 16 = 880$. That means that when $x = 2$, $y = 880$, or $880 = f(2)$, and that $(2, 880) = \big(2, f(2)\big)$ is in the relation.

The term function is also used to refer to the rule, that is the process or method, that associates with each real value of x in the domain of the function exactly one real value of y. An equation that is solved for the dependent variable y in terms of x might be called a function by formula (if it is single valued). If each step on the right side of the solved equation is a subrule yielding a single numeric value, the result for any given value of x in the domain is a single value of y. Frequently that single value of y is numerically calculable, for example when the right side of the equation consists of simple processes, such as algebraic operations. An example of such a function rule is $2x^2 + 7$, meaning $y = 2x^2 + 7$.