Transformations

Math 130 *Kovitz*

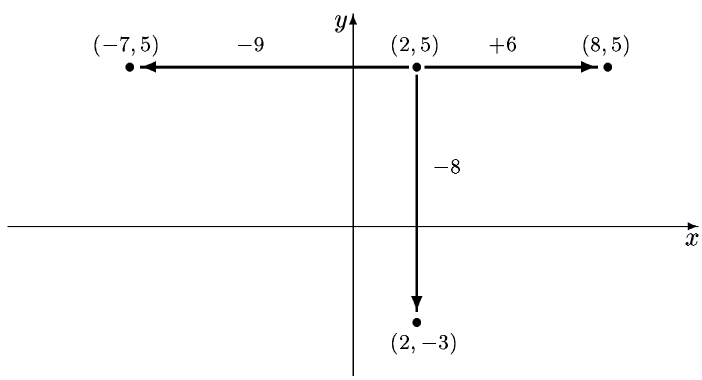
Geometric Transformations of a Point

To illustrate the types of transformations that we shall consider, we shall apply them to a curve consisting of a single point, for example the point $(2,5)$.

Translations

The first type of transformation is called a translation or a shift, a movement of the point a fixed distance in a specified direction. Although translations are defined for movement along lines in all directions in the plane, for simiplicity we shall consider only those in a horizontal or a vertical direction.

What happens if the point $(2,5)$ moves 6 units to the right? Clearly the new point is $(8,5)$. That is the result of adding 6 to the first coordinate, here 2. Similarly if it moved 9 units to the left, the new point $(-7,5)$ would be found by subtracting 9 from the first coordinate, here 2.

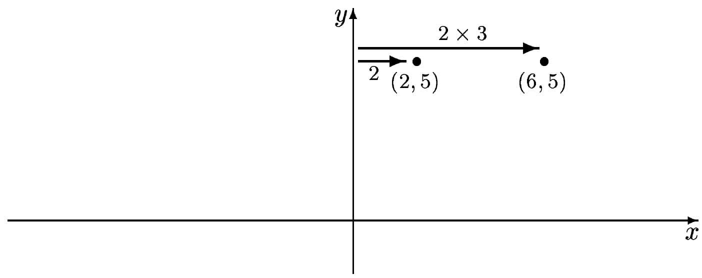


Similarly to move up a given number of units, one adds that value to the y-coordinate, and to move down a given number of units, one subtracts that value from the y-coordinate. For example the point $(2,5)$, when moved down 8 units ends up at $(2,-3)$. See the above diagram.

Expansions and Compressions

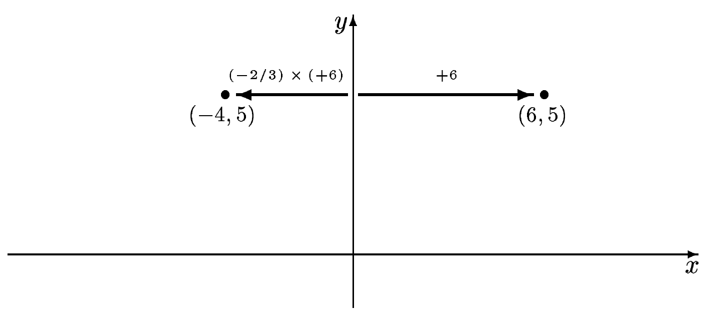
The second type of transformation is called either an expansion or stretch, or a compression or shrink. It is the result of multiplying the distance of the point from a fixed line by a constant. The only fixed lines we shall consider are $x = 0$ and $y = 0$ (the axes). Since in this case the distance of a point from the line $x = 0$ is simply its first coordinate and the distance of a point from the line $y = 0$ is simply its second coordinate, the case is somewhat analagous to translation, with multiplication of the coordinate instead of addition.

What happens if we multiply the x-coordinate of the point $(2,5)$ by 3? Clearly the point moves 3 times as far as before from the line $x = 0$ (the y-axis) and ends up with coordinates $(6,5)$. That is called an expansion or a stretch. This of it as pulling this point, and all other points on the curve being stretched, away from the axis as one would pull a rubber band to stretch it. Since the expansion applies to the first coordinate, the movement is in the x direction or away from the line $x = 0$ (the y-axis).



A multiplication factor less than 1 compresses or shrinks the distance of the point from the relevant axis. If we multiply the x-coordinate of the point $(2,5)$ by $1/5$, the point moves to a position $1/5$ as far from the line $x = 0$ (the y-axis) and ends up with coordinates $(2/5, 5)$. Think of it as pushing this point, and all other points on the curve being shrunk, toward the axis as one would compress a rubber ball to shrink its size. Since the compression applies to the first coordinate, the movement is in the x direction or toward the line $x = 0$ (the y-axis).

What happpens if the first coordinate of the point is multiplied by -1? That clearly reflects the point across the line $x = 0$ (the y-axis) because it changes the sign of the x-coordinate. Multiplying by any other negative constant will first expand or compress the distance and then reflect the result across the relevant axis. For example let us take the point $(6,5)$ and multiply the x-coordinate by $-2/3$. The result will be that the point is reflected across the line $x = 0$ and its distance from the line $x = 0$ is shrunk by a factor of $2/3$. The point will now have the coordinates $(-4,5)$.



Similarly if the *second* coordinate of a given point is multiplied by a positive constant, the distance of that point from the line $y = 0$ (the x-axis) is either stretched or shrunk according to whether the constant is greater than or less than 1. And, if the second coordinate is multiplied by a negative constant, the point is reflected across the line $y = 0$ (the x-axis) and the distance from the x-axis stretched or shrunk according to whether the absolute value of the constant is greater than or less than 1. For example if the constant is -7, we must look at it as $-1(7)$ which equals -1 times $|-7|$. Then the actions of the reflection and of the expansion or contraction can be viewed separately.

Transformations Determined from Equations

Translations of a Point

Let us consider the equation

$$(x+10)^2+(y-2)^2=25.$$

What does its graph look like? If a point (a,b) is on its graph, then we must have $(a+10)^2+(b-2)^2=25$. One way to find all the points is to try various ordered pairs to see which ones make the equation true. However, there is a much simpler method of graphing this equation based on the ideas of translations of points.

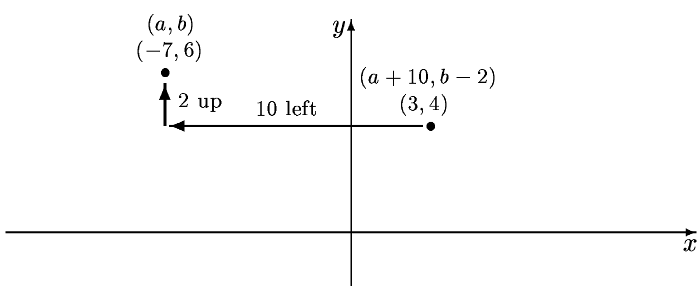
If we introduce new variables u and v with $u=x+10$ and $v=y-2$, then upon substitution we get

$$u^2+v^2=25,$$

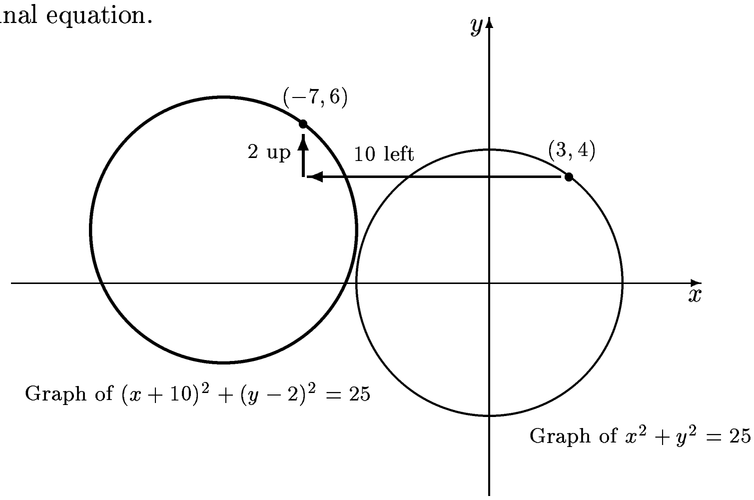
a familiar equation of which we readily know the graph (a circle with radius 5 and center at the origin).

Take any point (c,d) on the graph of $u^2+v^2=25$, say $(3,4)$. We have $3^2+4^2=25$. Now look at a solution (a,b) of the original equation in x and y. Since $(a+10)^2+(b-2)^2=25$, it must be true that $(a+10,b-2)$ is a point on the graph of $u^2+v^2=25$.

To get from the point $(a+10,b-2)$ to the desired point on the original graph we merely have to subtract 10 from $a+10$ and add 2 to $b-2$. The desired point on $(x+10)^2+(y-2)^2=25$ is found by taking the point on $u^2+v^2=25$ and moving it 10 to the left and 2 up. So any point (a,b) on the graph of the original equation can be obtained from a point on $u^2+v^2=25$ by two translations. Here is the picture. (Since $x^2+y^2=25$ has the same graph as $u^2+v^2=25$ we may graph $u^2+v^2=25$ as $x^2+y^2=25$ and locate the point $(3,4)$ on the x,y-axes.)

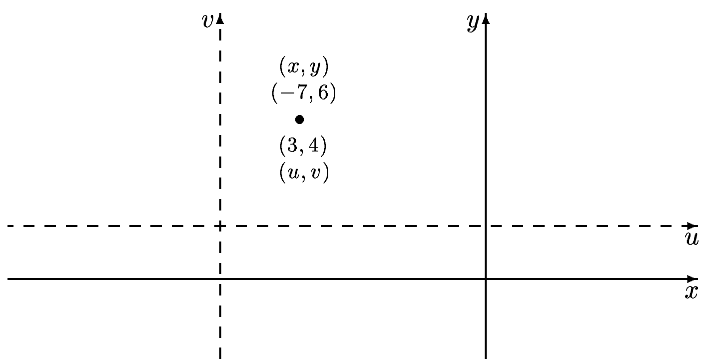


Starting with the known graph of $x^2+y^2=25$, we can get the desired graph by translating the entire graph 10 to the left and 2 up. Note that this movement is the *opposite* of the directions implicitly associated with the $+10$ and -2 in the original equation.

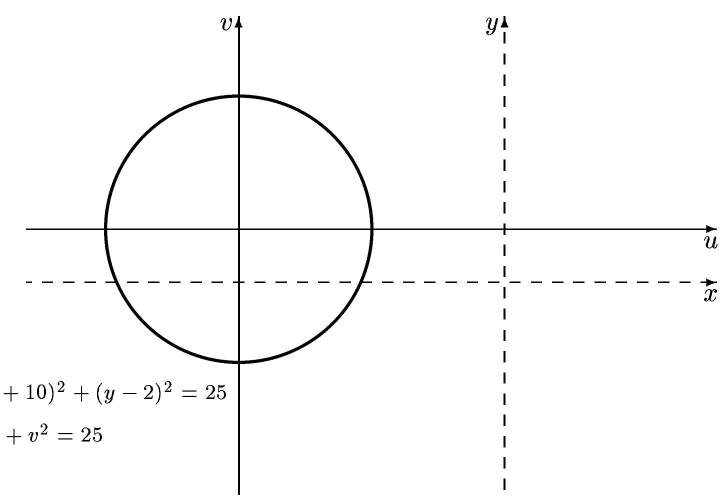


Translation of a Graph by Relocation of the Axes

An alternate theory of linear transformations involves redrawing the axes for the new variable, keeping the original points in their places. Suppose that the curve was drawn for the equation $(x+10)^2+(y-2)^2=25$ and then new variables $u=x+10$ and $v=y-2$ were introduced. Relative to the new line $u=0$ (the v-axis), the first coordinate of any point would have to be 10 greater than it was before. This could be accomplished by moving the vertical axis (formerly $x=0$ and now $u=0$) 10 to the *left*. Similarly, relative to the new line $v=0$ (the u-axis), the second coordinate of any point would have to be 2 less than it was before. This could be accomplished by moving the horizontal axis (formerly $y=0$ and now $v=0$) 2 *up*.



Another, more useful, way of looking at the same transformation is first to draw the graph of $u^2+v^2=25$ on the u,v-axes and then use $x=u-10$ and $y=v+2$ to relocate the coordinate axes. In this case the line $x=0$ (the y-axis) will have to be located 10 units to the right of the v-axis, and the line $y=0$ (the x-axis) will have to be located 2 units below the u-axis. Relative to the x and y axes we now have an accurate graph of the original equation. Compare this graph with respect to the x and y axes with the bottom graph on the previous page. (Here because of two operations, each opposite in direction, the movement of the axes is the *same* as the directions implicitly associated with the $+10$ and -2 in the original equation.)



Graph of $(x+10)^2+(y-2)^2=25$

Graph of $u^2+v^2=25$

Effects of Expansions and Compressions on the Location of a Point

Let us consider the equation

$$(3x)^2 + (y/2)^2 = 25.$$

What does its graph look like? If a point (a, b) is on its graph, then we must have $(3a)^2 + (b/2)^2 = 25$. One way, just as before, to find all the points is to try various ordered pairs to see which ones make the equation true. However, there is a much simpler method of graphing the equation based on the ideas of expansions and compressions of points.

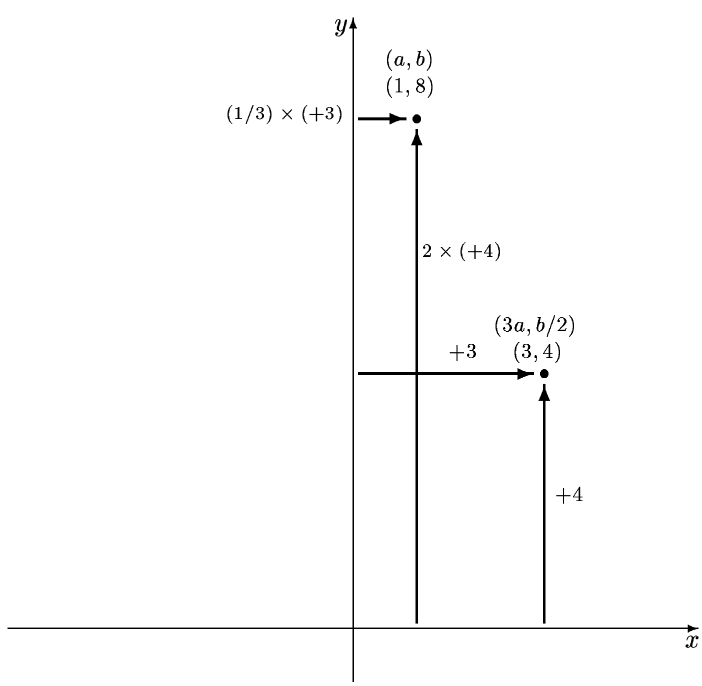
If we introduce new variables u and v with $u = 3x$ and $v = y/2$, then upon substitution we get

$$u^2 + v^2 = 25,$$

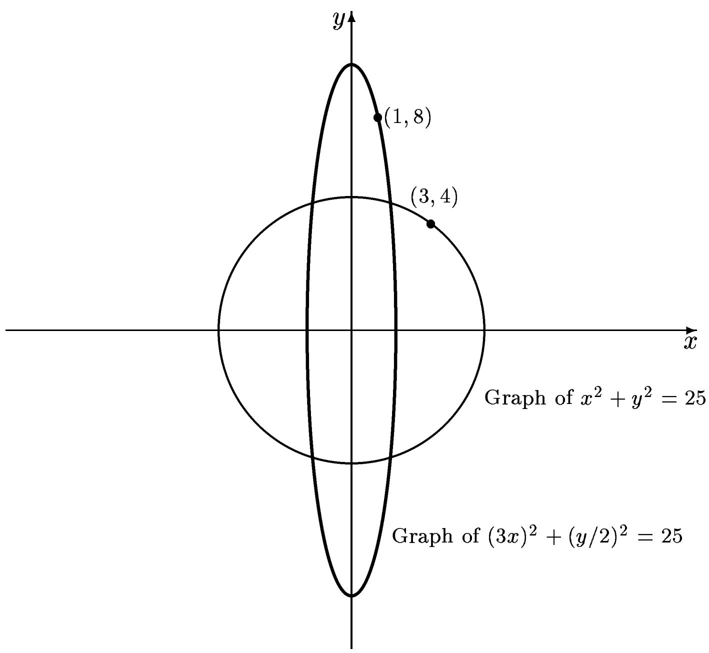
a familiar equation of which we readily know the graph (a circle with radius 5 and center at the origin).

Take any point (c, d) on the graph of $u^2 + v^2 = 25$, say $(3, 4)$. We have $3^2 + 4^2 = 25$. Now look at a solution (a, b) of the original equation in x and y. Since $(3a)^2 + (b/2)^2 = 25$, it must be true that $(3a, b/2)$ is a point on the graph of $u^2 + v^2 = 25$.

To get from the point $(3a, b/2)$ to the desired point on the original graph we merely have to divide $3a$ by 3 and multiply $b/2$ by 2. The desired point on $(3x)^2 + (y/2)^2 = 25$ is found by locating the point on $u^2 + v^2 = 25$; then compressing its distance from the line $x = 0$ (the y-axis) by a factor of $1/3$ toward the y-axis and expanding its distance from the line $y = 0$ (the x-axis) by a factor of 2 away from the x-axis. So, any point (a, b) on the graph of the original equation can be obtained from a point on $u^2 + v^2 = 25$ by one compression and one expansion. Here is the picture. (Since $x^2 + y^2 = 25$ has the same graph as $u^2 + v^2 = 25$ we may graph $u^2 + v^2 = 25$ as $x^2 + y^2 = 25$ and locate the point $(3, 4)$ on the x, y-axes.)



Starting with the known graph of $x^2+y^2=25$, we can get the desired graph by first compressing the entire graph by a factor of $1/3$ toward the line $x=0$ (the y-axis) and then expanding the entire graph by a factor of 2 away from the line $y=0$ (the x-axis). As in the case of translations we note that this compression or expansion is the *opposite* of the action apparently implied by the 3 and $1/2$ in the original equation. This graph incidentally has the form of an ellipse.



Here again there is an alternate theory of expansions and compressions that involves redrawing the scales of the axes. For example to get the graph of $(3x)^2+(y/2)^2=25$ one could rescale the x-axis renaming the physical position of the former unit length with the length $1/3$ and rescale the y-axis renaming the physical position of the former unit length with the length 2. This will result in a coordinate system which will not depict the true shapes of the curves which we graph. Occasionally, as in trigonometry, when we are not very concerned with a true-to-scale graph such a method will be useful. I shall leave it to you to explore it now if you wish.

Although no examples are now given, when each occurrence of x in an equation is replaced by a certain negative multiple of x, the result is to reflect the graph of the equation across the line $x=0$ (the y-axis) in addition to stretching or shrinking it. Similarly when each occurrence of y is replaced by a certain negative multiple of y, the result is to reflect the graph of the equation across the line $y=0$ (the x-axis) in addition to stretching or shrinking it.

Summary of Replacement Method

Given an equation whose graph is known and a second equation exactly the same as the first except that each occurrence of x has been *replaced* by $x \pm c$ (c a real constant), the second graph is exactly the same as the first except that the first graph has been translated along the x-axis (left or right) a distance equal to $|c|$ but in the *opposite* direction. The same holds for y with the movement being along the y-axis (up or down).

Given an equation whose graph is known and a second equation exactly the same as the first except that each occurrence of x has been *replaced* by x times k (k a non-zero constant), the second graph is exactly the same as the first except that the first graph has been compressed or expanded horizontally, with an action opposite to the one implied by the size of $|k|$. (That is, compression when $|k| > 1$ and expansion when $|k| < 1$.) Additionally if $k < 0$ then the graph is reflected across the y-axis (the line $x = 0$). The same holds for replacements of y with the expansion or compression being vertical and the reflection, if any, being across the x-axis (the line $y = 0$).

Functions

Transformations of Second Coordinate: Function Form

If one variable in the equation, usually y, is alone on the left side we say that the equation is in function form or that it is solved for y. In such a case we have an alternative way of looking at a transformation.

For example, take the graph of $(y - 2) = x^2$. This may be obtained from the known graph of $y = x^2$ by moving the graph of $y = x^2$ up 2. As we have seen before a replacement of all occurrences of y by $y - 2$ produces a movement in the *opposite* direction, here up 2.

If we solve the equation for y, we get $y = x^2 + 2$. Now looking at the value 2 as a number added to the *whole right side*, the translation in the y direction is up as the number 2 suggests. Do not confuse this with the replacement method. If there was a substitution, the effect of the translation, compression, or expansion is the opposite as would normally be expected. If there was an operation on the *entire* right side, the effect is the *same* as one would normally expect.

Transformations of First Coordinate: $f(x - h)$ and $f(x)$

We may look at $f(x - h)$ as a the result of a composition, or chain, of two functions. The first function is given by $g(x) = x - h$ and the second function is given by $f(x)$.

The rule for $f(x-h)$ is that $g(x) = x-h$ and $(f \circ g)(x) = f(g(x)) = f(x-h)$. For a given value a, g takes a to $a-h$ and then f takes $a-h$ to $f(a-h)$. In other words we first subtract h from a and then apply f to the resulting number.

Now suppose (a, b) is a point in the function f. Then $(a + h, b)$ is a point in the function $f(x - h)$. The proof of that is simple. We know that $b = f(a)$. When $a + h$ is substituted for x in $f(x - h)$, we get $f\big((a + h) - h\big) = f(a) = b$. That means that the second coordinate for the composite function with the rule $f(x - h)$ is b when the first coordinate is $a + h$. The point on the new graph representing the composite function is $(a+h, b)$, which is a horizontal translation of distance h in the opposite direction as one might initially expect by the minus sign separating x and h. The reason for this fact is similar to the reason for the reflected point being on the equation of the reflection. It is a compensation of the $+h$ for the $-h$ resulting in the same output for the function. That is, the inputs are different but the outputs are the same. When a constant is added to each x, the graph is translated horizontally.

Similar arguments, omitted here, apply to the graph of f with the rule $f(kx)$.