

Even and Odd Functions

Math 130 Kovitz

Definitions

Definition 1 A function is an **even function** if its graph is symmetric with respect to the y -axis.

Definition 2 A function is an **odd function** if its graph is symmetric with respect to the origin.

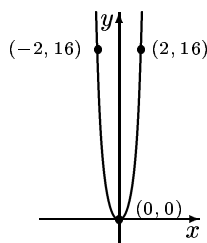
Even Functions

Suppose f is a function. Let $(a, f(a))$ be an ordered pair in the function. If the graph of $f(x)$ is symmetric with respect to the y -axis, then the reflection of the point $(a, f(a))$ across the y -axis is also on the graph of the function. Thus $(-a, f(a))$ is also a point on the graph of the function. But when $x = -a$, the second coordinate in the function f is equal to $f(-a)$, and there is only one possible second coordinate in the function when $x = -a$. Thus we conclude that $f(-a) = f(a)$ for any a in the domain. That is the same as saying that $f(-x) = f(x)$.

Often f is a function by formula. In that case if and only if f is even will $f(-x)$ and $f(x)$ have the same value for all x in the domain. A common method of showing that f is even is to show that for any x in the domain the expressions for $f(-x)$ and $f(x)$ are algebraically equivalent.

For example, to decide if the function f given by $f(x) = x^4$ is even, we must check if $f(-x) = f(x)$. Here we have $f(-x) = (-x)^4 = x^4 = f(x)$, so the function is even.

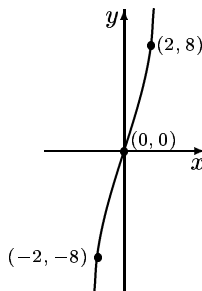
Note the graph:



Graphically it appears symmetric across the y -axis, so it is even. The fact that it is even could be established algebraically, without ever graphing it—this was just further confirmation of the fact.

For example, to decide if the function f given by $f(x) = x^3$ is even, we must check if $f(-x) = f(x)$. Here we have $f(-x) = (-x)^3 = -x^3 \neq x^3 = f(x)$, so the function is not even.

Note the graph:



Graphically it does not appear symmetric across the y -axis, so it is not even. This could also be shown by taking a point such as $(2, 8)$ and showing that its reflection across the y -axis is not on the graph. That would mean that the graph is not symmetric across the y -axis, hence not even.

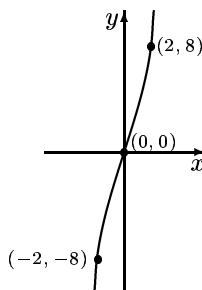
Odd Functions

Suppose f is a function. Let $(a, f(a))$ be a point on the function. If f is symmetric with respect to the origin, then the reflection of that point through the origin will also be on the function. Thus $(-a, -f(a))$ is also a point in the function. But when $x = -a$, the second coordinate on the function f is equal to $f(-a)$, and there is only one possible second coordinate on the function when $x = -a$. Thus we conclude that $f(-a) = -f(a)$ for any a in the domain. That is the same as saying that $f(-x) = -f(x)$.

Often f is a function by formula. In that case, if and only if f is odd will $f(-x)$ and $-f(x)$ have the same value for all x in the domain. A common method of showing that f is odd is to show that for any x in the domain the expressions for $f(-x)$ and $-f(x)$ are algebraically equivalent.

For example, to decide if the function f given by $f(x) = x^3$ is odd, we must check if $f(-x) = -f(x)$. Here we have $f(-x) = (-x)^3 = -x^3 = -f(x)$, so the function is odd.

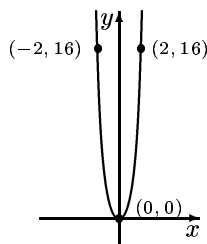
Note the graph:



Graphically it appears symmetric through the origin, so it is odd. The fact that it is odd could be determined algebraically, without ever graphing it—this was just further confirmation of the fact.

For example, to decide if the function f given by $f(x) = x^4$ is odd, we must check if $f(-x) = -f(x)$. Here we have $f(-x) = (-x)^4 = x^4 \neq -x^4 = -f(x)$, so the function is not odd.

Note the graph:



Graphically it is not symmetric through the origin, so it is not odd.

Input/Output Model

Let us consider a function f as a function machine with x as the input and $f(x)$ as the output. Then an even function is one where, whenever the sign of the input is changed, the output is unchanged. Similarly an odd function is one where, whenever the sign of the input is changed, the sign of the output is also changed.

For example, consider the function f where the input is the velocity of an automobile in meters per second, with a negative value indicating travel in a reverse direction, and the output is the speedometer reading in miles per hour. This is an even function since travel at an equal speed in a negative direction leads to the same speedometer reading. With a special speedometer that also shows negative readings, the resulting function g would be an odd function since travel at an equal speed in a negative direction would lead to a speedometer reading with the opposite sign.

Algebraic Procedure to Test if a Function is Odd, Even, or Neither

Given the formula $f(x)$ for a function f , first find $f(-x)$.

- If $f(-x)$ and $f(x)$ are algebraically equivalent, then the function is even.

If not, find $-f(x)$.

- If $f(-x)$ and $-f(x)$ are algebraically equivalent, then the function is odd.
- If neither is the case, then the function is neither even nor odd. Be careful—perhaps they are actually algebraically equivalent but you couldn't show it. The definitive way to show a function is neither even nor odd is to find a value a for x so that $f(-a) \neq f(a)$ and a value b for x so that $f(-b) \neq -f(b)$.

The following chart might be useful:

$$\begin{array}{ccccc} f(x) & \longleftrightarrow & f(-x) & \longleftrightarrow & -f(x) \\ & \text{COMPARE} & & \text{COMPARE} & \end{array}$$

For example, to decide if the function f given by $f(x) = 2x^3 - x^5$ is even or odd, first find $f(-x) = 2(-x)^3 - (-x)^5 = -2x^3 + x^5$.

- Since $f(-x) = -2x^3 + x^5$ and $f(x) = 2x^3 - x^5$ are not algebraically equivalent, the function is not even.

Now find $-f(x) = -(2x^3 - x^5) = -2x^3 + x^5$.

- Since $f(-x) = -2x^3 + x^5$ and $-f(x) = -2x^3 + x^5$ are identical, the function is odd.

For example, to decide if the function f given by $f(x) = x - 1$ is even or odd, compare $f(-x)$ with $f(x)$ and $f(-x)$ with $-f(x)$.

$$\begin{array}{ccccc} f(x) = x - 1 & \longleftrightarrow & f(-x) = -x - 1 & \longleftrightarrow & -f(x) = -x + 1 \\ & \text{COMPARE} & & \text{COMPARE} & \end{array}$$

Since neither comparison involves equivalent expressions, the function given by $x - 1$ is neither even nor odd.

For example, to decide if the function f given by $f(x) = (1+x^2)^3$ is odd, even, or neither, compare $f(-x)$ with $f(x)$ and $f(-x)$ with $-f(x)$. We have $f(-x) = (1+(-x)^2)^3 = (1+x^2)^3$. Since $f(-x) = f(x)$ the function f is even.

General Comments

A relation whose formula has all x appearing to even powers is symmetric across the y -axis and one that has all y appearing to even powers is symmetric across the x -axis. One that has both all x and all y appearing to even powers is symmetric through the origin.

There is no special designation for a function which is symmetric across the x -axis because such a function could not have two different values for y and still be a function. Only the trivial case of the function f given by $f(x) = 0$ for all x in the domain would fit such a designation.