Completing the Square

Math 130 *Kovitz*

Given a quadratic function in general form

$$ax^2 + bx + c \quad (\text{with } a \neq 0),$$

we wish to put the function in standard form

$$a(x-h)^2 + k.$$

Steps in completing the square for quadratic polynomials

1. Isolate the constant term to get:

$$ax^2 + bx + \qquad c.$$

2. If $a \neq 1$, factor out the coefficient of the x^2 term from the x^2 and x-terms, giving

$$a(x^2 + \tfrac{b}{a}x) + c.$$

   To find the coefficient of x after the factoring, simply divide bx by a, and b/a will be the new coefficient.

   To catch common errors in algebra at this step it is a good idea to check your factoring by expanding the parentheses using the distributive law.

   It is not a good idea to factor the a out of the constant; it will just lead to confusion later and it serves no purpose.

3. Replace the parentheses by square brackets. This will warn you of later pitfalls and ensure a correct solution.

$$a[x^2 + \tfrac{b}{a}x] + c.$$

$\frac{b}{a}x$	The x-term
$\frac{b}{a}$	Dropped the x.
$\frac{b}{2a}$	Divided by 2.
$\frac{b^2}{4a^2}$	Squared it.

4. Find the constant that completes the square by
   (a) Halving the new coefficient of the x term. This number is the second term inside the perfect square we will later find. It is equal to $b/2a$.
   (b) Squaring the result—this is the number that completes the square (it is always positive). It equals $b^2/4a^2$.

5. Add and subtract the completing number *within the square brackets*. Now there are four terms within the brackets. Call the 4th term—which is always negative—the dangling term. The quadratic is $a[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}] + c$.

6. Rewrite the first three terms as a perfect square, $(x + \text{second term})^2$. Leave the dangling term alone. The quadratic is now $a[(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2}] + c$.

7. Distribute the constant that is outside the square brackets. Be sure to distribute to the dangling term. Result: $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$.

8. Combine the resulting constants to get a single number, which will be k.
   Result: $a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$.

9. Since the standard form requires $x - h$, the value of h is minus the value of the second term: $h = -\frac{b}{2a}$. Think of the quadratic as $a\left(x - (-\frac{b}{2a})\right)^2 + \left(c - \frac{b^2}{4a}\right)$.

Summary of the Steps

- Isolate the constant term.
- Factor out the front number from the x^2 and x-terms. The inside x term is the orginal x term divided by the front number.

  Do not factor the front number out of the constant.
- Replace the parentheses with square brackets.
- Start with the x-term and 1) remove the x; 2) divide by two; 3) square the result.
- Inside the brackets, add and subtract the result of 3).
- Using the variable added to the result of 2), replace the first three terms with the perfect square, which is the square of that binomial. The binomial must first be enclosed in parentheses, then squared.
- Distribute the outside constant to the perfect square term *and* to the dangling term.
- Combine the constants.

Numeric Example of Completing the Square

It is far simpler when dealing with actual numbers.

$-4x^2 + 6x - 3$ will be used as an example.

$-4x^2 + 6x \qquad - 3$

Factoring out the -4 gives an inside x term of $\dfrac{6x}{-4}$, which reduces to $-\dfrac{3}{2}x$.

$-4\left(x^2 - \frac{3}{2}x\right) - 3$

$-4\left[x^2 - \frac{3}{2}x\right] - 3$

$-\frac{3}{2}x$	The x-term.
$-\frac{3}{2}$	Dropped the x.
$-\frac{3}{4}$	Divided by 2.
$\frac{9}{16}$	Squared it.

$-\frac{3}{2}x$ becomes $-\frac{3}{2}$ becomes $-\frac{3}{4}$ becomes $\frac{9}{16}$.

The second number of the binomial is $-\frac{3}{4}$; the completing number is $\frac{9}{16}$.

$-4\left[x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right] - 3$

The variable on the first line of the box plus the third line of the box is the binomial whose square replaces the first three terms.

The first three terms will be replaced by $\left(x + (-\frac{3}{4})\right)^2 = (x - \frac{3}{4})^2$.

$-4\left[\left(x - \frac{3}{4}\right)^2 - \frac{9}{16}\right] - 3$

The brackets remind us that distributing the -4 needs to be done carefully.

The key step follows:

$-4\left(x - \frac{3}{4}\right)^2 + \frac{9}{4} - 3$ The result of distributing to the dangling term is $-4\left(-\frac{9}{16}\right) = \frac{9}{4}$.

$-4\left(x - \frac{3}{4}\right)^2 - \frac{3}{4}$.

Finally, be careful: h is $\frac{3}{4}$, not $-\frac{3}{4}$ because the formula asks for $x - h$. But k is $-\frac{3}{4}$.

Vertex Formulas

Since this a quadratic equation, the direct formulas for the vertex apply. The technique of completing the square is important because it is useful in other cases, besides quadratic functions, but is a good idea to check the previous result with the formulas.

$a = -4$, $b = 6$, and $c = -3$.

$h = -\frac{b}{2a} = -\frac{6}{-8} = \frac{-6}{-8} = \frac{3}{4}$. That checks.

$k = c - \frac{b^2}{4a} = -3 - \frac{36}{-16} = -3 + \frac{36}{16} = -3 + \frac{9}{4} = -\frac{3}{4}$. That checks also.

Problems (Complete the squares and, if you wish, multiply out the results.)

1. $x^2 - 4x + 5$ Answer: $h = 2$ and $k = 1$.
2. $2x^2 - 4x + 5$ $a = 2$, $h = 1$, and $k = 3$.
3. $-x^2 - 4x + 5$ $a = -1$, $h = -2$, and $k = 9$.
4. $3x + 1 - x^2$ $a = -1$, $h = 3/2$, and $k = 3\frac{1}{4}$.
5. $1 - 6x - 2x^2$ $a = -2$, $h = -3/2$, and $k = 5\frac{1}{2}$.
6. $-9x^2 + 12x + 7$ $a = -9$, $h = 2/3$, and $k = 11$.
7. $\frac{5}{9}x^2 + 10x + 43$ $a = 5/9$, $h = -9$, and $k = -2$.
8. $-.06x^2 + .84x - 4.24$ $a = -.06$, $h = 7$, and $k = -1.30$.
9. $rx^2 - 2sx + t$ $a = r$, $h = s/r$, and $k = t - s^2/r$.
10. $(m + 3n)x^2 + 4x + \dfrac{4}{m + 3n}$ $a = m + 3n$, $h = -2/(m + 3n)$, and $k = 0$.
11. $-\frac{1}{3}x^2 - 3x + 2$ $a = -\frac{1}{3}$, $h = -\frac{9}{2}$, and $k = \frac{35}{4}$.
12. $x^2 + x + 1$ $a = 1$, $h = -\frac{1}{2}$, and $k = \frac{3}{4}$.
13. $2x^2 - 7$ $a = 2$, $h = 0$, and $k = -7$. It was given in standard form, nothing to do.