Completing the Square

Math 130 *Kovitz* 2012

Quadratic Polynomials: Desired Form

Given a quadratic polynomial

$$ax^2 + bx + c \quad (\text{with } a \neq 0),$$

we wish to find numbers h and k such that

$$ax^2 + bx + c = a(x - h)^2 + k.$$

Think of this as finding, for a polynomial that is given in standard form, an algebraically equivalent expression in a certain form. That form is as the sum of a constant multiple of the square of x minus a constant and another constant. The right side is called a completed square because the x appears only as part of a term to be squared. Also note that when the expression is considered as a function, $f(x)$, the right side is obtained from the function $g(x) = x^2$ by applying one stretch or shrink, by a factor of a in the y direction, and two shifts, one in the x direction by h and one in the y direction by k.

The left side of the above equation has exactly one term of degree 2 in x, namely ax^2, and the right side of the above equation, by expanding the square to $a(x^2 - 2hx + h^2) + k$ or $ax^2 - 2ahx + ah^2 + k$, also has exactly one term of degree 2 in x. It is therefore impossible for the expressions to be algebraically equivalent unless the same constant, a, multiplies the squared term, $(x - h)^2$.

Considering

$$ax^2 + bx + c = ax^2 - 2ahx + ah^2 + k$$

and setting equal the terms of degree 1 in x, we conclude that $b = -2ah$ or $h = -b/2a$.

By setting equal the constant terms from both sides, we can also conclude that $c = ah^2 + k$ or $k = c - ah^2$. Substituting $-b/2a$ for h, we conclude that $k = c - ah^2 = c - a(-b/2a)^2 = c - a(b^2/4a^2) = c - b^2/4a$.

We now know that in all cases when $a \neq 0$ such an expression $a(x - h)^2 + k$ can be found from a, b, and c by certain formulas. Although we now have a solution, an alternative approach offers more insight into the problem.

Basic Techniques for Completing the Square

Given the formula

$$x^2 + Bx,$$

we wish to find an equivalent formula of the form

$$(x + q)^2 + k.$$

For example, take x^2+14x. What values of q and k would lead to an algebraically equivalent expression $(x + q)^2 + k$?

Look at $(x + 9)^2$, which equals $x^2 + 18x + 81$. Here from the given $x^2 + 18x$ we would need to find a number q such that $(x + q)^2 = x^2 + 18x + \cdots$. How is the value q related to 18? When we square $x + 9$, the x term, $18x$, has a coefficient exactly twice the number 9. Going from $18x$ to the constant term, q, of the original expression, $x + q$, just divide by 2 the coefficient 18. Furthermore to find the third term of the perfect square just square q (this term will always be positive).

We thus conclude that $x^2 + 18x$ are the first two terms of the perfect square $x^2 + 18x + 81$.

Now returning to the example $x^2 + 14x$ we find that the expression $(x + q)^2 + k$ must begin with $(x + 7)^2$. To keep the expression equivalent one could simply add to $x^2 + 14x$ the expression $49 - 49$ or zero, which doesn't affect equality. Then $x^2 + 14x + 49 - 49 = (x^2 + 14x + 49) - 49 = (x + 7)^2 - 49$. This process, where $(x^2 + 14x + 49) - 49$ is produced from $x^2 + 14x$, is the actual completion of the square.

The process may be summarized in the following diagram:

$$x^2 + Bx \xrightarrow{(\ /2)^2} \quad = \quad \underbrace{x^2 + Bx + \left(\frac{B}{2}\right)^2}_{\downarrow} \overbrace{\ - \left(\frac{B}{2}\right)^2}^{\text{these cancel}}$$

$$= \quad \left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2$$

Application of Completing the Square to Quadratic Polynomials

We return to the problem of finding for a given quadratic polynomial

$$ax^2 + bx + c \quad (\text{with } a \neq 0)$$

values h and k such that

$$ax^2 + bx + c = a(x - h)^2 + k.$$

It is instructive to classify the cases according to the values of a and c.

Case 1 $(b = 0)$

We are given

$$ax^2 + c.$$

The expression is already in the desired form, with $h = 0$ and $k = c$.

Case 2 $(a = 1 \text{ and } c = 0)$

We are given

$$x^2 + bx.$$

For example, take $x^2 + 10x$. What values of h and k would lead to an algebraically equivalent expression $(x - h)^2 + k$?

We find that the expression $(x - h)^2 + k$ must begin with $(x + 5)^2$. To maintain equivalence just add to $x^2 + 10x$ the expression $25 - 25$ or zero, which doesn't affect equality. Then $x^2 + 10x + 25 - 25 = (x^2 + 10x + 25) - 25 = (x + 5)^2 - 25$.

This process, where $(x^2 + 10x + 25) - 25$ is produced from $x^2 + 10x$, is the actual completion of the square for $x^2 + Bx$ with $B = b$. The equivalent form $(x + q)^2 + k$ is $(x + 5)^2 - 25$; and in this example where $B = b = 10$ we get $q = 5$ and $k = -25$.

The result is that $h = -5$ and $k = -25$. Note that h and q, the constant inside the completed square, always have opposite signs. Think of $(x + 5)$ as $\big(x - (-5)\big)$.

Further examples:

$x^2 + 2x = x^2 + 2x + 1 - 1 = (x + 1)^2 - 1$ Ans: $h = -1$ and $k = -1$.

$x^2 - 4x = x^2 - 4x + 4 - 4 = (x - 2)^2 - 4$ Ans: $h = 2$ and $k = -4$.

Case 3 ($a = 1$, $b \neq 0$, and $c \neq 0$)

We are given

$$x^2 + bx + c.$$

For example, take $x^2+14x-11$. Carry the constant c at the end of the expression and apply the process of the previous case to the first two terms, obtaining $(x^2 + 14x + 49 - 49) - 11$. We get $(x + 7)^2 - 60$.

The result is that $h = -7$ and $k = -60$.

Another example:

$$x^2 - 6x + 5 = (x^2 - 6x) + 5 = (x^2 - 6x + 9 - 9) + 5 = (x - 3)^2 - 4$$

Ans: $h = 3$ and $k = -4$.

Case 4 ($a \neq 1$, $c = 0$)

We are given

$$ax^2 + bx.$$

For example, take $3x^2 - 24x$. First factor out a, here 3, to get $3[x^2 - 8x]$. Next apply Case 2 inside the brackets. They multiply through by a, here 3.

We get $3[x^2 - 8x + 16 - 16] = 3[(x - 4)^2 - 16] = 3(x - 4)^2 - 48$. *Be very careful* to apply the distributive law correctly and multiply by 3 the -16 also.

The result is that $a = 3$, $h = 4$, and $k = -48$.

Another example:

$$3x^2 - 6x = 3[x^2 - 2x] = 3[x^2 - 2x + 1 - 1] = 3[(x - 1)^2 - 1] = 3(x - 1)^2 - 3$$

Ans: $h = 1$ and $k = -3$.

Case 5 (In general)

We are given

$$ax^2 + bx + c.$$

For example, take $-5x^2 + 45x - \frac{287}{4}$. First factor out the 5 out of the first two terms only and then proceed as in the previous case, carrying the constant c.

We get

$$\begin{aligned} -5[x^2 - 9x] - \tfrac{287}{4} &= -5[x^2 - 9x + (\tfrac{9}{2})^2 - (\tfrac{9}{2})^2] - \tfrac{287}{4} \\ &= -5[(x - \tfrac{9}{2})^2 - \tfrac{81}{4}] - \tfrac{287}{4} = -5(x - \tfrac{9}{2})^2 - 5(-\tfrac{81}{4}) - \tfrac{287}{4} \\ &= -5(x - \tfrac{9}{2})^2 + \tfrac{405}{4} - \tfrac{287}{4} = -5(x - \tfrac{9}{2})^2 + \tfrac{118}{4} = -5(x - \tfrac{9}{2})^2 + \tfrac{59}{2}. \end{aligned}$$

The result is that $a = -5$, $h = \frac{9}{2}$, and $k = \frac{59}{2}$. (Technically, $(\frac{9}{2})^2$ is really $(\frac{-9}{2})^2$.)

Another example:

$$\begin{aligned} 3 + 4x - 2x^2 &= -2[x^2 - 2x] + 3 = -2[x^2 - 2x + 1 - 1] + 3 = -2[(x - 1)^2 - 1] + 3 \\ &= -2(x - 1)^2 - 2(-1) + 3 = -2(x - 1)^2 + 2 + 3 = -2(x - 1)^2 + 5 \end{aligned}$$

Ans: $a = -2$, $h = 1$, and $k = 5$.

As a check let us now multiply out the answer, getting

$$-2(x - 1)^2 + 5 = -2(x^2 - 2x + 1) + 5 = -2x^2 + 4x + 3.$$

In the general case we have $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$ with $B = \frac{b}{a}$.

This leads to

$$a\left[\left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2\right] + c = a\left[\left(x + \left(\frac{b/a}{2}\right)\right) - \left(\frac{b/a}{2}\right)^2\right] + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c = a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b^2}{4a^2}\right) + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

This is the same as the earlier desired form since $h = -q = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$.

Uses of the Completed Square Form of a Quadratic Polynomial

1. Graphing.

The vertex is (h, k). The sign of a determines whether the parabola opens up or down. The absolute value of a indicates the shape, the larger the value of a the narrower the curve, that is the greater the stretch of the graph of $y = x^2$ needed to produce the graph of the quadratic polynomial.

2. Finding the zeros of the quadratic function.

We may find the zeros of the quadratic function from the completed square version using the following procedure.

$a(x - h)^2 + k = 0.$

$a(x - h)^2 = -k.$

$(x - h)^2 = -\frac{k}{a}.$

$x - h = \pm\sqrt{-\frac{k}{a}}.$

$x = h \pm \sqrt{-\frac{k}{a}}.$

Substituting $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$, we get

$$\begin{aligned} x &= -\frac{b}{2a} \pm \sqrt{-\frac{c - \frac{b^2}{4a}}{a}} = -\frac{b}{2a} \pm \sqrt{-\frac{(4ac - b^2)/4a}{a}} = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\ &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

Actually to find the zeros of a quadratic function a slightly simpler application of completing the square is available.

When the polynomial $ax^2 + bx + c$ is set equal to zero, it is often better if one first divides the terms by a.

$x^2 + (b/a)x + (c/a) = 0.$

$x^2 + (b/a)x = -c/a.$ (here $B = b/a$)

$x^2 + \left(\frac{b/a}{2}\right)x + \left(\frac{b/a}{2}\right)^2 - \left(\frac{b/a}{2}\right)^2 = -\frac{c}{a}.$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}.$$

$$x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \pm\frac{\sqrt{b^2-4ac}}{2a}.$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Steps in completing the square for quadratic polynomials

- Put the x^2 and the x terms in parentheses, *not including* the constant term.
- If $a \neq 1$, factor out the coefficient of the x^2 term from the parentheses. Some examples are.

$$-2x^2 + 14x = -2(x^2 - 7x)$$

$$12x^2 + 8x = 12(x^2 + \tfrac{2}{3}x)$$

$$-\tfrac{1}{2}x^2 - 16x = -\tfrac{1}{2}(x^2 + 32x)$$

$$.74x^2 - .37x = .74(x^2 - .5x)$$

  To catch common errors in algebra at this step it is a good idea to check your factoring by expanding the parentheses using the distributive law. For example: $12(x^2 + \frac{2}{3}x) = 12x^2 + 12(\frac{2}{3}x) = 12x^2 + 8x$.

- Find the constant that completes the square by
  (a) Halving the coefficient, b/a, of the x term and obtaining $q = -h$.
  (b) Squaring the result—this is the number that completes the square; call it h^2 (it is always positive).
- Add and subtract h^2 *within the parentheses.*
- Bring the negative of the completing number out of the parentheses by multiplying it by a (an application of the distributive law of multiplication). The result will be $-ah^2$. That value is added to the original constant.
- Rewrite the parentheses as a perfect square. The sign of the number q obtained in step (a) (which is the number added to or subtracted from x here) will be the opposite of the sign of h. If desired the term inside the perfect square may be rewritten as $(x - h)$.

Problems (Complete the squares and, if you wish, multiply out the results.)

1. $x^2 - 4x + 5$ Answer: $h = 2$ and $k = 1$.
2. $2x^2 - 4x + 5$ $a = 2$, $h = 1$, and $k = 3$.
3. $-x^2 - 4x + 5$ $a = -1$, $h = -2$, and $k = 9$.
4. $3x + 1 - x^2$ $a = -1$, $h = 3/2$, and $k = 3\frac{1}{4}$.
5. $1 - 6x - 2x^2$ $a = -2$, $h = -3/2$, and $k = 5\frac{1}{2}$.
6. $-9x^2 + 12x + 7$ $a = -9$, $h = 2/3$, and $k = 11$.
7. $\frac{5}{9}x^2 + 10x + 43$ $a = 5/9$, $h = -9$, and $k = -2$.
8. $-.06x^2 + .84x - 4.24$ $a = -.06$, $h = 7$, and $k = -1.30$.
9. $rx^2 - 2sx + t$ $a = r$, $h = s/r$, and $k = t - s^2/r$.
10. $(m + 3n)x^2 + 4x + \dfrac{4}{m + 3n}$ $a = m + 3n$, $h = -2/(m + 3n)$, and $k = 0$.