Completing the Square

This is a procedure that *starts* with a quadratic polynomial

$$ax^2 + bx + c$$

and *ends* with an expression of the form

$$a(x + h)^2 + k,$$

which is algebraically *equal* to the starting polynomial. We say that $a(x + h)^2 + k$ is the "completed square form" of $ax^2 + bx + c$.

Completing the square is a procedure that changes the *form* of a quadratic polynomial. It is useful for solving quadratic equations, graphing parabolas, and other applications. The essential feature of the completed-square form is that there is *no linear term*, that is, no term of the form, 'number times variable.' The polynomial $ax^2 + bx + c$ has linear term bx, but completing the square removes the linear term. $a(x + h)^2 + k$ has quadratic term $a(x + h)^2$ and constant term k.

For example, we may see directly that $x^2 + 2x + 1 = (x + 1)^2$; then $(x + 1)^2$ is the completed square form of $x^2 + 2x + 1$, with $a = 1$, $h = 1$, $k = 0$. But when direct pattern recognition doesn't serve, we need a method.

Method

1. Basic operation: Given $x^2 + bx$,

$$\text{write } \left(x + {}^{b}/_{2}\right)^2 - \left({}^{b}/_{2}\right)^2.$$

That is, do "x plus ${}^{b}/_{2}$, then square, then subtract $\left({}^{b}/_{2}\right)^2$."

The 2 expressions are algebraically equal:

$$\begin{aligned}\left(x + {}^{b}/_{2}\right)^2 - \left({}^{b}/_{2}\right)^2 &= x^2 + 2x\left({}^{b}/_{2}\right) + \left({}^{b}/_{2}\right)^2 - \left({}^{b}/_{2}\right)^2 \\ &= x^2 + bx.\end{aligned}$$

The point is that what you just wrote, $\left(x + {}^{b}/_{2}\right)^2 - \left({}^{b}/_{2}\right)^2$, is the completed square form of $x^2 + bx$ (with $a = 1$, $h = {}^{b}/_{2}$, $k = -\left({}^{b}/_{2}\right)^2$.)

Examples. Complete the squares.

$$x^2 + 2x = \left(x + {}^{2}/_{2}\right)^2 - \left({}^{2}/_{2}\right)^2 = (x + 1)^2 - 1 \qquad \text{(compare example above)}$$

$$x^2 + 4x = \left(x + {}^{4}/_{2}\right)^2 - \left({}^{4}/_{2}\right)^2 = (x + 2)^2 - 4$$

$$x^2 - 6x = \left(x + {}^{(-6)}/_{2}\right)^2 - \left({}^{(-6)}/_{2}\right)^2 = (x - 3)^2 - 9$$

$$x^2 - 5x = \left(x - {}^{5}/_{2}\right)^2 - {}^{25}/_{4}$$

2. Next, to complete the square of $x^2 + bx + c$, do the basic operation on $x^2 + bx$, and carry the c.

Example. Complete the square.

$$\begin{aligned}x^2 - 6x + 3 &= [x^2 - 6x] + 3 \\ &= [(x - 3)^2 - 9] + 3 \\ &= (x - 3)^2 - 6.\end{aligned}$$

3. Finally, to complete the square of $ax^2 + bx + c$, factor a from the x terms:

$$ax^2 + bx + c = a[x^2 + \left({}^{b}/_{a}\right)x] + c,$$

do the basic operation inside the [ ], then simplify.

Examples. Complete the assorted squares.

$$\begin{aligned} x^2 - 8x + 1 &= (x-4)^2 - 4^2 + 1 \\ &= (x-4)^2 - 15 \end{aligned}$$

$$\begin{aligned} 2x^2 - 8x + 1 &= 2[x^2 - 4x] + 1 \\ &= 2[(x-2)^2 - 4] + 1 \\ &= 2(x-2)^2 - 2\cdot 4 + 1 \\ &= 2(x-2)^2 - 7 \end{aligned}$$

$$\begin{aligned} 3x^2 + 6x - 1 &= 3[x^2 + 2x] - 1 \\ &= 3[(x+1)^2 - 1] - 1 \\ &= 3(x+1)^2 - 3 - 1 \\ &= 3(x+1)^2 - 4 \end{aligned}$$

$$\begin{aligned} -x^2 + 6x - 1 &= -[x^2 - 6x] - 1 \qquad (a = -1\text{; be careful}) \\ &= -[(x-3)^2 - 9] - 1 \\ &= -(x-3)^2 + 9 - 1 \\ &= -(x-3)^2 + 8 \end{aligned}$$

$$\begin{aligned} -3x^2 + 6x - 1 &= -3[x^2 - 2x] - 1 \\ &= -3[(x-1)^2 - 1] - 1 \\ &= -3(x-1)^2 + 3 - 1 \\ &= -3(x-1)^2 + 2 \end{aligned}$$

$$\begin{aligned} ax^2 + bx + c &= a\left[x^2 + \left({}^{b}/_{a}\right)x\right] + c \qquad (\text{the general case}) \\ &= a\left[\left(x + \left({}^{b}/_{2a}\right)\right)^2 - \left({}^{b}/_{2a}\right)^2\right] + c \\ &= a\left(x + \left({}^{b}/_{2a}\right)\right)^2 - a\left({}^{b}/_{2a}\right)^2 + c \\ &= a\left(x + \left({}^{b}/_{2a}\right)\right)^2 - {}^{b^2}/_{4a} + c \end{aligned}$$

Practice. Complete the squares. You can always check an answer by squaring and collecting terms: you should get the original polynomial.

1. $x^2 + 4x + 5$ $\quad = (x+2)^2 + 1$
2. $x^2 - 4x + 5$ $\quad = (x-2)^2 + 1$
3. $x^2 + 7x - 3$ $\quad = (x + {}^{7}/_{2})^2 - 15\,{}^{1}/_{4}$
4. $2x^2 - 4x + 5$ $\quad = 2(x-1)^2 + 3$
5. $-x^2 - 4x + 5$ $\quad = -(x+2)^2 + 9$
6. $3x + 1 - x^2$ $\quad = -(x - {}^{3}/_{2})^2 + 3\,{}^{1}/_{4}$
7. $3 + 4x - 2x^2$ $\quad = 5 - 2(x-1)^2$