Finding Vertices of Parabolas

Math 130 *Kovitz*

An equation of the form $y = ax^2 + bx + c$ defines a function of x, given by $f(x) = ax^2 + bx + c$.

To graph the equation, yielding a parabola if $a \neq 0$, we complete the square and end up with $y = a(x-h)^2 + k$. The vertex is then (h, k). Note that $a(x-h)^2 + k$ will have its minimum value of k (if $a > 0$) or its maximum value of k (if $a < 0$) when $x - h = 0$, since for other values of x, $a(x-h)^2 + k$ equals k plus a positive multiple of a.

Also note that a will be the same in the completed square form as in the original equation. This is true because in the completed square form when the term $(x-h)$ is squared, the first term of the result is simply x^2. That must then be multiplied by a constant to yield ax^2 as the result. So the constant has to be a.

If $a > 0$, the parabola opens upward and has a minimum y-value of k.

If $a < 0$, the parbola opens downward and has a maximum y-value of k.

As an example, done by completing the square the long way, we have

$$y = x^2 + 14x + 43 = (x^2 + 14x + 49 - 49) + 43 = (x+7)^2 - 6.$$

A Shortcut Way to Find h and k

Step 1: $h = -b/2a$

The proof is omitted but it uses long completing the square beginning with $y = ax^2 + bx + c$.

For example, take $y = x^2 + 14x + 43$. We have $h = -b/2a = -14/(2(1)) = -14/2 = -7$.

The equation will be $y = (x-h)^2 + k = (x - (-7))^2 + k = (x+7)^2 + k$.

Step 2: $k = f(h)$

A simple way to find k is developed by noting that when $x = h$, we have $y = a(h-h)^2 + k = a(0)^2 + k = k$. We can go back to the original equation, set $x = h$, the value of which was already found in step 1, and find y. *That value of y will be k.* So, in the case of a parabola of the form $y = ax^2 + bx + c$, we have $k = f(h)$. This method does not apply to all uses of completing the square, only to parabolas of this form.

For example, continue the previous example of $y = x^2 + 14x + 43$ with $h = -7$. We get $k = f(h) = f(-7) = (-7)^2 + 14(-7) + 43 = 49 - 98 + 43 = -6$. The final completed square form is now $y = (x+7)^2 - 6$. (In this substitution of h into f there is a further check that the second term in the intermediate calculations will always be -2 times the first term.)

Yet another way to find k is to note that $ax^2 + bx + c = a(x + \frac{b}{2a})^2 + k$. If we multiply out the right-hand side, we see that $ax^2 + bx + c = ax^2 + bx + \frac{b^2}{4a} + k$, leading to the conclusion that $k = c - \frac{b^2}{4a}$. (See the next page for the computation.)

Alternate Ways of Obtaining k

As alternatives to step 2 there are formulas for k:

$$k = c - \frac{b^2}{4a} \quad \text{or} \quad k = c - ah^2$$

For the previous example,

$$k = c - \frac{b^2}{4a} = 43 - \frac{14^2}{4(1)} = 43 - \frac{196}{4} = 43 - 49 = -6$$

or

$$k = c - ah^2 = 43 - 1(7)^2 = 43 - 49 = -6.$$

The most frequent use of these alternative methods of finding k is to find k without finding h when h is not required. We often see that in max-min problems.

For example, given $y = 3x^2 + 36x + 117$, find k. We have

$$k = c - \frac{b^2}{4a} = 117 - \frac{36^2}{4(3)} = 117 - \frac{36(36)}{12} = 117 - 36(3) = 117 - 108 = 9.$$

Thus $k = 9$.

Further Examples of the Shortcut Method and Alternate Ways of Obtaining k

Example 1: Given $y = 2x^2 - 4x + 9$, we have

$$h = \frac{-(-4)}{2(2)} = \frac{4}{4} = 1$$

and

$$k = f(h) = f(1) = 2(1)^2 - 4(1) + 9 = 7.$$

The completed square form is then $y = 2(x-1)^2 + 7$.

Example 2: Given $y = 1 - 6x - 2x^2$, we have

$$h = \frac{-(-6)}{2(-2)} = \frac{6}{-4} = -\frac{3}{2}$$

and

$$k = f(h) = f(-3/2) = 1 - 6(-3/2) - 2(-3/2)^2 = 1 + 9 - 9/2 = 11/2.$$

The completed square form is then $y = (x + 3/2)^2 + 11/2$.

Example 3: Given $y = x^2 - 30x$, we have

$$h = \frac{-(-30)}{2} = \frac{30}{2} = 15$$

and

$$k = f(h) = f(15) = 15^2 - 30(15) = 225 - 450 = -225.$$

Note as a check that -2 times 225, the first term, equals -450, the second term.

The completed square form is then $y = (x - 15)^2 - 225$.

In this example to obtain k directly without finding h, use

$$k = c - \frac{b^2}{4a} = 0 - \frac{(-30)^2}{4(1)} = -\frac{900}{4} = -225,$$

getting $k = -225$.