Max-min Examples

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Procedures in Max-min Problems

1. Assign variables. Make sure to include the assignment of a variable to the quantity being maximized or minimized. Later on we will try to make that variable a function of one of the other variables.

2. Find a formula, a function of the variables, for the quantity to be maximized or minimized.

3. Reduce the formula, whenever possible, to one variable (if it is not already in one variable). By solving equation(s) express the other variable(s) in terms of a single variable. (In all of the present examples the equations to be solved for in this step are linear, in general they need not be linear.)

   Then substitute the results in the formula. Now consider the formula as a function, f, of that one variable, which is usually x.

   A particular assignment of variables will often simplify the resulting computations. Feel free to begin the problem again and assign the variable x to a different quantity if that might make things easier.

   (Such a problem, where these linear constraints add enough information to express the formula in terms of a single variable, is not unusual.)

4. If f is a quadratic function, its graph will be a parabola. Find the vertex using algebraic techniques on that function. The second coordinate of the vertex will be the maximum or minimum value of the function. There are always shortcut methods; they are often preferable, especially when only h or only k is needed. Otherwise complete the square.

   In the completed square version, $f(x) = a(x-h)^2 + k$, $f(x)$ is thus expressed as a variable part, $a(x-h)^2$, plus a fixed part, k.

   If $a > 0$, $f(x)$ is least when the variable part, $a(x-h)^2$, is zero since $a(x-h)^2$ is always zero or more. That is, $f(x)$ is least when $x = h$, and then $f(x) = k$. So $f(x)$ has a minumum value when $x = h$, and that minimum value is $f(h) = k$. The ordered pair with the minimum value of f is (h, k). The corresponding point (h, k) is the vertex of the graph.

   If $a < 0$, $f(x)$ is greatest when the variable part, $a(x-h)^2$, is zero since $a(x-h)^2$ is always zero or less. That is, $f(x)$ is greatest when $x = h$, and then $f(x) = k$. So $f(x)$ has a maximum value when $x = h$, and that maximum value is $f(h) = k$. The ordered pair with the maximum value of f is (h, k). The corresponding point (h, k) is the vertex of the graph.

5. Once h, the value of x, is found, find the values of the other variables at the vertex (where the quantity under consideration has a maximum or minimum value). Use the same linear equations which reduced the formula to one variable and substitute h for x.

   If k, the second coordinate of the vertex, was not already found by parabola techniques, find the desired maximum or mimimum value of the given quantity, which was to be maximized or minimized, by substituting these values into the original formula for that quantity.

Example 1

Two numbers add to 12 and have the largest possible product. What are they? What is their product?

1. Assignment of variables:

   Let x = first number.
   Let z = second number.
   Let p = the product to be maxmized.

2. Formula for the quantity to be maximized:

   $p = xz$

3. Reduction to one variable, using an equation relating the two variables:
   (Although the relationship in this example is linear, in general it need not be.)

   $x + z = 12$
   $z = 12 - x$

   Now substitute $12 - x$ for z in the formula $p = xz$. Since the resulting equation expresses p as a quadratic polynomial in x, we may look at p as a function of x and use the notation $p = f(x)$.

   $p = xz = x(12 - x) = 12x - x^2 = -x^2 + 12x$
   $p = f(x) = -x^2 + 12x$ (a quadratic function of x)

4. Finding the vertex (here a maximum point, since from $f(x) = -x^2 + 12x$ we get $a = -1 < 0$, meaning that the graph opens down):

   $f(x) = -x^2 + 12x = -(x^2 - 12x) = -(x^2 - 12x + 36 - 36) =$
   $-(x^2 - 12x + 36) + 36 = -(x - 6)^2 + 36$
   (The shortcut methods could also be used.)

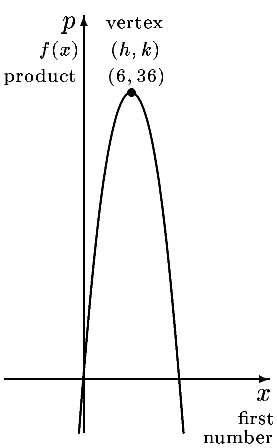
   The vertex is (6,36).
   If $x = 6$, $f(x) = 36$; otherwise $f(x) = 36 -$ some positive quantity.

   At the vertex, $p = f(x) = 36$ and $x = 6$. That means that the product of the two numbers is 36, a maximum, when the first number is equal to 6.

   A shortcut method is: $h = \frac{-b}{2a} = \frac{-12}{2(-1)} = 6$ (using $y = -x^2 + 12x$)

   $k = f(h) = f(6) = -(6)^2 + 12(6) = -36 + 72 = 36$

   The vertex is $(h, k) = (6, 36)$.



5. Conversion to solutions of the variables x and z, asssigned in the initial step:

   At the maximum point, the vertex, we have:
   $x = 6$.
   $z = 12 - x = 12 - 6 = 6$.
   The first number $= x = 6$.
   The second number $= z = 6$.

   Their product $= p = xz = 6 \cdot 6 = 36$. It is also equal to k, the maximum value of $f(x)$, found as the second coordinate of the vertex.

Example 2

Two numbers add to 12 and when twice the square of one of them is added to the other, the result is the minimum possible. What are the two numbers?

1. Assignment of variables:

   Let x = first number.
   Let w = second number.
   Let r = result.

2. Formula for the quantity to be maximized:

   $r = 2x^2 + w$

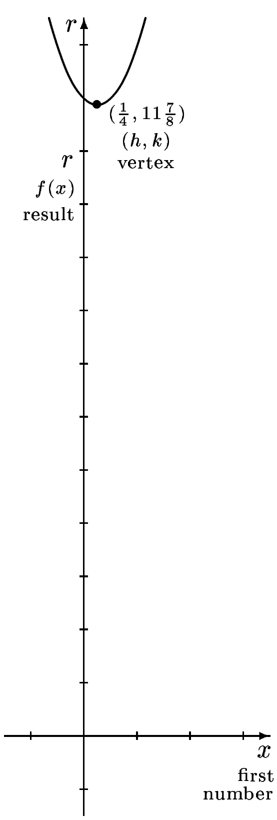
3. Reduction to one variable, using an equation relating the two variables:
   (Although the relationship in this example is linear, in general it need not be.)

   $x + w = 12$
   $w = 12 - x$

   Now substitute $12-x$ for w in the formula $r = 2x^2+w$. Since the resulting equation expresses r as a quadratic polynomial in x, we may look at r as a function of x and use the notation $r = f(x)$.

   $r = 2x^2 + w = 2x^2 + (12 - x) = 2x^2 - x + 12$
   $r = f(x) = 2x^2 - x + 12$ (a quadratic function of x)

   Since $a = 2 > 0$, the graph opens up and has a minimum point.



4. Finding h, the x-value of the vertex (k, the minimum value of the result is not required here):

   $$h = \frac{-b}{2a} = \frac{-(-1)}{2(2)} = \frac{1}{4}$$ (Long completing the square could also be used.)

   Long completing the square: $f(x) = 2x^2 - x + 12 = 2(x^2 - \frac{1}{2}x) + 12$
   $= 2(x^2 - \frac{1}{2}x + \frac{1}{16} - \frac{1}{16}) + 12 = 2(x - \frac{1}{4})^2 - 2(\frac{1}{16}) + 12 = 2(x - \frac{1}{4})^2 + 11\frac{7}{8}$
   So $h = \frac{1}{4}$ and $k = 11\frac{7}{8}$.
   The vertex is $(\frac{1}{4}, 11\frac{7}{8})$.
   If $x = \frac{1}{4}$, $f(x) = 11\frac{7}{8}$; otherwise $f(x) = 11\frac{7}{8}$ – some positive quantity.

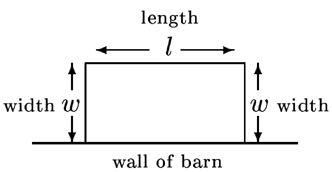
5. Conversion to solutions of the variables x and w, assigned in the initial step:

   At the minimum point, the vertex, we have:
   $x = \frac{1}{4}$.
   $w = 12 - x = 12 - \frac{1}{4} = 11\frac{3}{4}$.
   The first number $= x = \frac{1}{4}$.
   The second number $= w = 11\frac{3}{4}$.

   This problem is a curiosity. At the minimum it might seem reasonable to conclude that x, the part of 12 that is being squared and then doubled, equals zero and w, the part merely being added, equals 12. Such an answer makes the square of the first number as small as possible.
   Then: $2(0)^2 + 12 = 0 + 12 = 12$.
   Here: $2\left(\frac{1}{4}\right)^2 + 11\frac{3}{4} = \frac{2}{16} + 11\frac{3}{4} = 11\frac{7}{8}$, which is less.

Example 3

A farmer builds a rectangular pen inside his barn with 60 feet of fencing, using a wall of the barn for one side of the pen. Find the maximum area that he can enclose.



1. Assignment of variables:

   Let w = width of the pen.
   Let l = length of the pen (assume that side l is parallel to the wall.)
   Let a = area.

2. Formula for the quantity to be maximized:

   $a = lw$

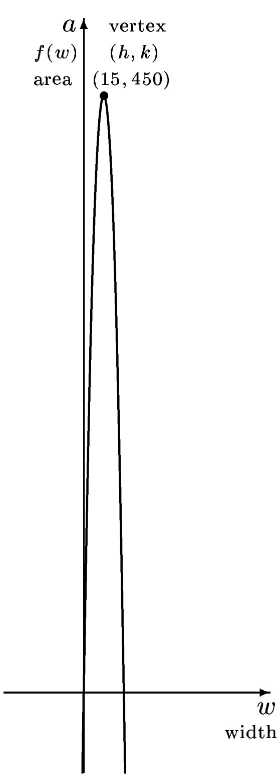
3. Reduction to one variable, using an equation relating the two variables:
   (Although the relationship in this example is linear, in general it need not be.)

   $l + 2w = 60$
   $l = 60 - 2w$

   Now substitute $60 - 2w$ for l in the formula $a = lw$. Since the resulting equation expresses a as a quadratic polynomial in w, we may look at a as a function of w and use the notation $a = f(w)$.

   $a = lw = w(60 - 2w) = 60w - 2w^2 = -2w^2 + 60w$
   $a = f(w) = -2w^2 + 60w$ (a quadratic function of w)

   Since $a = -2 < 0$, the graph opens down and has a maximum point.



4. Finding the k of the vertex (h, the value of the width that makes the area a maximum, is not required here):

   $k = c - \frac{b^2}{4a} = 0 - \frac{60^2}{4(-2)} = -\frac{3600}{-8} = \frac{-3600}{-8} = 450$ (Long completing the square could also be used.)

   Long completing the square: $f(w) = -2w^2 + 60w = -2(w^2 - 30w)$
   $= -2(w^2 - 30w + 225 - 225) = -2(w-15)^2 - 2(-225) = -2(w-15)^2 + 450$
   So $h = 15$ and $k = 450$.

   The vertex is $(15, 450)$.
   If $w = 15$, $f(w) = 450$; otherwise $f(w) = 450 -$ some positive quantity.

5. Conversion to solutions of the variables l and w assigned in the initial step: (this is not required—*the problem is solved*—but let us do it as a check.)

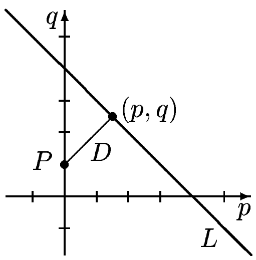
   At the maximum point, the vertex, we have:

   $w = h = \frac{-b}{2a} = \frac{-60}{2(-2)} = \frac{-60}{-4} = 15.$

   $l = 60 - 2w = 60 - 2(15) = 60 - 30 = 30.$
   The width of the pen $= w = 15$ feet.
   The length of the pen $= l = 30$ feet.

   The area $= a = lw = 15 \cdot 30 = 450$ square feet. It is also equal to k, the maximum value of $f(w)$, found as the second coordinate of the vertex.

Example 4



Let L be the line with equation $p + q = 4$ and let P be the point $(0, 1)$. Find the point on L closest to P, and find the distance between P and that point.

1. Assignment of variables:

   Let (p, q) be a typical point of L. Then

   $p =$ the first coordinate of a point on the line;

   $q =$ the second coordinate of a point on the line;

   $D =$ the distance between P and (p, q).

2. Formula for the quantity to be maximized:

   We seek the point (p, q) on L for which D has the least value. Now D is least whenever D^2 is least, and the latter is more convenient to use because it does not involve the square root of a complicated polynomial. We can express D^2 in terms of p and q by

   $$D^2 = p^2 + (q - 1)^2.$$

   (We used the distance formula with $p_1 = 0$, $p_2 = p$, $q_1 = 1$, and $q_2 = q$.)

3. Reduction to one variable, using an equation relating the two variables:

   (Although the relationship in this example is linear, in general it need not be.)

   When a point (p, q) is on L, its coordinates satisfy $p + q = 4$, the equation defining L. We can rewrite this as $q = 4 - p$ and substitute to get D^2 as a quadratic function of p alone.

   Now substitute $4 - p$ for q in the formula $D^2 = p^2 + (q - 1)^2$. Since the resulting equation expresses D^2 as a quadratic polynomial in p, we may look at D^2 as a function of p and use the notation $D^2 = f(p)$.

   $D^2 = p^2 + (q - 1)^2 = p^2 + \left((4 - p) - 1\right)^2 = p^2 + (3 - p)^2 = 2p^2 - 6p + 9$

   $D^2 = f(p) = 2p^2 - 6p + 9$ (a quadratic function of p)

   Since $a = 2 > 0$, the graph opens up and has a minimum point.

   We seek the p for which this quadratic function has the least value.

4. Finding the vertex by completing the square:

   $f(p) = 2p^2 - 6p + 9 = 2(p^2 - 3p) + 9 = 2\left(p^2 - 3p + (\frac{3}{2})^2 - (\frac{3}{2})^2\right) + 9 =$
   $2(p - \frac{3}{2})^2 - 2(\frac{9}{4}) + 9 = 2(p - \frac{3}{2})^2 - \frac{9}{2} + 9 = 2(p - \frac{3}{2})^2 + \frac{9}{2}$

   So $h = \frac{3}{2}$ and $k = \frac{9}{2}$. (The shortcut methods could also be used.)

   The vertex is $(\frac{3}{2}, \frac{9}{2})$.

   We have $D^2 = f(p) = \frac{9}{2}$ when $p = \frac{3}{2}$. That means that the square of the distance is $\frac{9}{2}$, a minimum, when the first coordinate of the point is $\frac{3}{2}$.

   Once we have completed the square, D^2 is thus expressed as a variable part, $2(p - \frac{3}{2})^2$, plus a fixed part, $\frac{9}{2}$, and D^2 is least when the variable part is zero since the variable part is always zero or larger. That is, D^2 is least when $p = \frac{3}{2}$, and then $D^2 = \frac{9}{2}$.

5. Conversion to solutions of the variables p, q, and D, assigned in the initial step:

   At the minimum point, the vertex, we have:
   $p = \frac{3}{2}$
   $q = 4 - p = 4 - \frac{3}{2} = \frac{5}{2}$.
   The point on the line L we require is $(\frac{3}{2}, \frac{5}{2})$.

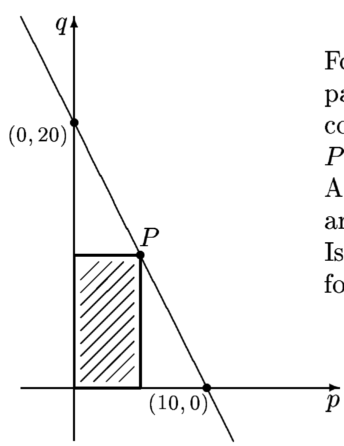
   The value k, the second coordinate of the vertex, which represents the minimum value of D^2, is equal to $\frac{9}{2}$.

   $D = \sqrt{D^2} = \sqrt{\frac{9}{2}} = 3/\sqrt{2}$

   The distance D between $(0, 1)$ and P, the closest point on the line, is $3/\sqrt{2} \approx 2.121$. To doublecheck, take the points $(0, 1)$ and $(\frac{3}{2}, \frac{5}{2})$ and apply the distance formula to them. The distance will be approximately 2.121, as previously found.

Exercise

In problems 1–3, for the given L and P, find the point on L closest to P, and find the distance between P and that point.

1. L: $2p + q = 3$ $\quad P = (1, 0)$ $\quad$ Ans. $(7/5, 1/5)$; $\sqrt{5}/5$
2. L: $p + 2q = 1$ $\quad P = (2, 2)$ $\quad$ Ans. $(1, 0)$; $\sqrt{5}$
3. L: $p - 2q = 1$ $\quad P = (0, 3)$ $\quad$ Ans. $(7/5, 1/5)$; $7/\sqrt{5}$
4. L: $3p - q = 8$ $\quad P = (0, 2)$ $\quad$ Ans. $(3, 1)$; $\sqrt{10}$
5. Find the point on the line $2p + q = 20$ closest to $(0, 0)$. $\quad$ Ans. $(8, 4)$
6. 

   For each point P on the line $2p + q = 20$, which passes through the points $(0, 20)$ and $(10, 0)$, consider the rectangle with upper right corner P and bounded by the axes, as in the sketch. Among these rectangles, find the one of largest area.
   Is the P of that rectangle the same point as was found in Problem 4?
   Ans. the rectangle with $P = (5, 10)$; no