Inverses

Math 130 *Kovitz*

Definition of the Inverse of a Relation

The inverse of a relation is the set of ordered pairs obtained by switching the first and second coordinates of the ordered pairs of the relation.

So the domain of the inverse will be the same as the range of the original, and the range of the inverse will be the same as the domain of the original. Be careful: sometimes two relations seem to be inverses of each other, but since the domains and ranges do not match up, they really are not. For example, consider $f(x) = \sqrt{x}$ and $g(x) = x^2$, with both functions defined over their complete domains.

Graph of the Inverse Relation

The graph of the inverse is the reflection across the line $y = x$ of the graph of the original relation.

Here are three ways that this reflection can be accomplished:

- Reflection of the curve across the line $y = x$. Often difficult to envision directly, it may instead be obtained by rotating the curve 90 degrees counterclockwise, and then flipping the resulting curve across the y-axis.
- Reflecting points of the original curve across the line $y = x$, by switching the first and second coordinates and replotting. If enough are reflected, a trend will often allow ust to draw the new curve through those points. It is always true that the graph of the orignal relation and the graph of the inverse relation have the same size and shape. Using that will often enable us to draw the new curve after a few points are reflected.
- (When the original relation is defined by an equation) Write the equation of the reflection across the line $y = x$ (the inverse) by switching the x and y in the equation of the orginal relation. Then graph this equation.

Determining if the Inverse of a Relation is a Function

If the original relation is one-to-one, meaning that no two x's in the original relation have the same value of y, then the inverse relation will be a function. This may be determined visually by the Horizontal Line Test: if two distinct points on the graph of the original relation lie on the same horizontal line, the inverse relation will not be a function.

If the relation is increasing everywhere or decreasing everywhere and the graph is connected, it is surely one-to-one and there is an inverse function.

Notation for the Inverse of a Function

If the inverse relation of the function f is also a function, we denote this new function by f^{-1}.

(Be careful: f^{-1} will not refer to the reciprocal function of f; the reciprocal of a function must be denoted by $1/f$.)

Using the definition of the inverse function, it always follows that when $f(a) = b$, $f^{-1}(b) = a$.

Before applying this rule, one should first verify that the function f is one-to-one.

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Finding a Formula for f^{-1}, the Inverse Function, given a formula for f

Replace $f(x)$ by y. (At this point we are on f and y is the output being determined by this formula from the input x.)

Switch x and y in the equation. (This will produce a formula for the reflection of the curve across the line $y = x$. That's a formula for f^{-1}, the inverse of f.)

Solve for y, if it is possible. (We need to express the output in terms of the input.)

Replace y by $f^{-1}(x)$. (Being on f^{-1}, the output y is the function f^{-1} of the input x.)

The special case where the original function is represented by a linear equation with slope $m \neq 0$ will have an inverse with slope $1/m$. From this we see that if the original line is increasing, the inverse line will also be increasing. This is part of a more general phenomenon: strictly increasing functions have inverse functions and the inverse functions are strictly increasing as well. (the same for strictly decreasing)

Finding a Formula for f^{-1}, the Inverse Function, given that $f(A) = B$

Same as above, except that there is no need to switch A and B in the equation. In the given formula for f, A is designated as the input and B is designated as the output. Just designate A as the output of f^{-1} and solve for A in terms of B. That solution is $f^{-1}(B)$.

The Composition of a Function with its Inverse

$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = a$ for any a in the domain of f, and

$(f \circ f^{-1})(b) = f(f^{-1}(b)) = b$ for any b in the domain of f^{-1}.

Proof: $a \xrightarrow{f} b \xrightarrow{f^{-1}} a$ and $b \xrightarrow{f^{-1}} a \xrightarrow{f} b$.

When f and g are functions such that $(g \circ f)(a) = g(f(a))$ for all a in the domain of f and $(f \circ g)(b) = f(g(b))$ for all b in the domain of g, it is established that f and g are inverse functions of each other.

Symmetry across the Line $y = x$ (a self-inverting function)

If the graph of the original relation happens to be symmetric across the line $y = x$, the original and the inverse will be equivalent. That means they will have exactly the same set of points, the same graph, and the same solution set if there is an equation. This cannot happen unless the domain and range of the orginal relation were the same.

If $y = x$ symmetry is to be present for a function, $(f \circ f)(a) = f(f(a))$ must be equal to a for all a in the domain of f. Sometimes that can easily be verified.

Inverse of a Composition String

When a function is the composition of two other functions, to get the inverse of the composition, take the inverse of each of the two other functions in the reverse order. This may be extended to a composition of three or more components.

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Decomposing a function into a verbal composition string will sometimes lead to an easier derivation of a formula for f^{-1} than the earlier method.

The Implicit Definition of the Inverse Function of f (assuming that f is one-to-one)

$f^{-1}(a)$ is the unique value w such that $f(w) = a$. This will be useful when a solution for y in the earlier method is not accessible to us.

When f^{-1} has no formula, because we have no way to solve for y after switching x and y, it cannot be evaluated directly. However, if we can guess a real number w, so that w input to the function f gives a, it follows that the number guessed is in fact $f^{-1}(a)$. This might have to be done for each value of a separately. Even if it cannot be determined, a unique value of w is known to exist.

Concept of the Inverse Function

The function f does something; the inverse function f^{-1} *undoes* it.