

Concepts of an Inverse

Math 130 Kovitz

Definition of the Inverse of a Relation

The inverse of a relation is the set of ordered pairs obtained by switching the first and second coordinates of the ordered pairs of the relation. What is the inverse of the inverse relation?

So the domain of the inverse will be the same as the range of the original, and the range of the inverse will be the same as the domain of the original. Be careful: sometimes two relations seem to be inverses of each other, but since the domains and ranges do not match up, they really are not. For example, consider $f(x) = \sqrt{x}$ and $g(x) = x^2$, with both functions defined over their complete domains.

Graph of the Inverse Relation

The graph of the inverse is the reflection across the line $y = x$ of the graph of the original relation.

From now on, consider only functions, not just relations.

When will the Inverse Relation turn out to be a Function?

If for every member of the range (every output) of the original function, there is exactly one input, the function is said to be one-to-one.

In that case the inverse is a function. If the original function is named f , its inverse function is designated by f^{-1} .

The inverse takes each output of the original function back to its input. So the input of the inverse function is a member of the range of the original, and its output is a unique member of the domain of the original function. In the aggregate, the inverse function takes the range of the original back to its domain. The inverse function is defined only over the range (outputs) of the original function.

Whatever the original function does, the inverse undoes it.

Examples

1. For each verbal function, decide if the inverse is a function. If so, find the inverse, in verbal form.
 - (a) Double the number.
 - (b) Add 3 to the number.
 - (c) Square the number.
2. A linear function is defined by the equation $y = 2x - 5$, and its graph contains the points (6, 7) and (10, 15).
 - (a) Will the inverse be a linear function. Why is that obvious?
 - (b) Find two points for the graph of the inverse, basing your choice on the two given points.
From these two points, find an equation of the inverse.
 - (c) Graph on the same axes the original function and the line $y = x$. From these two lines and their point of intersection, roughly graph the inverse.
 - (d) A simpler method to find the equation: First take any value a in the domain of the original function f . Note that f takes a to $2a - 5$.
Now we know that f^{-1} takes $2a - 5$ as its input and yields a as its output. So for this inverse, $2a - 5$ is x and a is y . That leads to the equation:

$$2y - 5 = x.$$

Solving for y , an equation for f^{-1} is found to be $y = \frac{x + 5}{2}$.

- (e) There's a very easy verbal method also (see next page: Composition String).

A Procedure for Finding an Equation of the Inverse Function from an Equation of the Original Function.

- Replace $f(x)$ by y .
- Switch x and y in this equation.
- Solve for y , if possible.
- Replace y with $f^{-1}(x)$.

The Composition of a Function with its Inverse

$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = a$ for any a in the domain of f , and

$(f \circ f^{-1})(b) = f(f^{-1}(b)) = b$ for any b in the domain of f^{-1} .

Proof: $a \xrightarrow{f} b \xrightarrow{f^{-1}} a$ and $b \xrightarrow{f^{-1}} a \xrightarrow{f} b$.

When f and g are functions such that $(g \circ f)(a) = g(f(a)) = a$ for all a in the domain of f and $(f \circ g)(b) = f(g(b)) = b$ for all b in the domain of g , it is established that f and g are inverse functions of each other.

Symmetry across the Line $y = x$ (relations or functions)

If the graph of the original is to be symmetric across the line $y = x$, the original and the inverse will be equivalent. That means they will have exactly the same set of points, the same graph, and the same solution set if there is an equation. If symmetry is to be present for a function, $(f \circ f)(a) = f(f(a))$ must be equal to a for all a in the domain of f . Sometimes that can easily be verified.

Inverse of a Composition String

When a function is the composition of two other functions, to get the inverse of the composition, take the inverse of each of the two other functions in the reverse order. This may be extended to a composition of three or more components.

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Decomposing a function into a verbal composition string will sometimes lead to an easier derivation of a formula for f^{-1} than the earlier method.

Example 2 part (e): Applying this technique to example 2 leads to a rather simple method of finding the inverse function.

Rewrite the function as: double the number then subtract 5.

The inverse must be the composition of the inverses of the subfunctions, in the opposite order. It is: add 5 then divide by 2. The formula is $\frac{x+5}{2}$. That's all we needed to do.

Example 3: Revisit example 2 and its methods for the fn. $f(x) = \sqrt{x+2} + 5$.

The Implicit Definition of the Inverse Function of f (assuming that f is one-to-one)

$f^{-1}(a)$ is the unique value w such that $f(w) = a$. This will be useful when it proves to be impossible to solve for y in the earlier method.

When f^{-1} has no formula, because it is not possible to solve for y after switching x and y , it cannot be evaluated directly. However, if we can guess a real number w , so that w input to the function f gives a , it follows that the number guessed is in fact $f^{-1}(a)$. This might have to be done for each value of a separately. Even if it cannot be determined, a unique value of w is known to exist.

Concept of the Inverse Function

The function f does something; the inverse function f^{-1} *undoes* it.