Function Defined by Pairs Problem

Math 130

Suppose that a relation is fully defined by the following four ordered pairs:

$$\left(\frac{1}{8}, 3\right), \left(\frac{1}{3}, \frac{1}{8}\right), (3, 8), \left(8, \frac{1}{3}\right).$$

1. Prove that the relation is a function. What are its domain and range? Call that function f.

2. Is the function symmetric across the line $y = x$?

3. Show that the function is one-to-one and therefore has an inverse function.

4. How are the reciprocal of f (the function $\frac{1}{f}$) and f^{-1} related?

5. Find a simple formula for $f(f(a))$ that holds for all a in the domain of f.

6. How are these four numbers related: $f(3)$, $f^{-1}(3)$, $f\left(\frac{1}{3}\right)$, $\frac{1}{f(3)}$?

7. Find a formula for $f\left(\frac{1}{a}\right)$ in terms of $f(a)$.

8. Prove that $f(f(f(f(a)))) = a$ for all a in the domain of f.

9. Draw arrow diagrams for $f(f(f(f(3))))$ and for $f^{-1}(f^{-1}(f^{-1}(f^{-1}(8))))$. Does the first answer support the assertion in part 8?

10. Does $f(a) = a$ have any solutions for this function f?

11. Explain why f cannot be represented by a simple algebraic formula.

Answers follow.

Answers.

1. Its graph passes the vertical line test.

   The domain and the range are the same. Each consists of the four real numbers: $\frac{1}{8}$, $\frac{1}{3}$, 3, and 8.

2. No. The point $(3, 8)$ is in the function, but its symmetric partner $(8, 3)$ is not.

3. Its graph passed the horizontal line test.

4. They are the same function.

5. $f(f(a)) = \frac{1}{a}$.

6. The last three numbers are equal.

7. $f\left(\frac{1}{a}\right) = \frac{1}{f(a)}$.

8. $f \circ f$ gives the recipocal of the input. If you take the composition of $f \circ f$ with itself, it must therefore give the identity.

9. $3 \to 8 \to \frac{1}{3} \to \frac{1}{8} \to 3$ and $8 \to 3 \to \frac{1}{8} \to \frac{1}{3} \to 8$. In part 8, we showed that the composition of f with itself four times is the identity. The arrows from 3 led to 3, confirming the identity.

10. No.

11. The reciprocal function cannot be represented by the composition of a simple algebraic function with itself. If the function f were to take one over the input, doing it twice would get back to the original number, not to the reciprocal. This is a bit informal, but seems reasonable.