Inverse Function Problems

Math 130 *Kovitz*

1. Let $f(x) = \dfrac{x-3}{4}$.

   Find $f^{-1}(x)$, the easiest way possible.

   Then find $f(11)$, $f^{-1}(11)$, $f^{-1}(f(11))$, and $f(f^{-1}(11))$.

   It is of interest to determine whether f is its own inverse (whether f is symmetric across the line $y = x$).

   Several approaches will be presented, with questions regarding their application:

   - Graphic.

     Graph f and f^{-1} on the same coordinate plane.
     True or false? Since the resulting picture is symmetric across the line $y = x$, that proves that this f is symmetric across the line $y = x$. If you answer false, explain the correct visual, graphic method to establish symmetry across the line $y = x$.
     True or false? Every time you draw the graphs of f and f^{-1} in the same coordinate system, you create a picture that is symmetric across the line $y = x$.

   - Function as an input-output model.

     Take any real number a. Apply f to it and then apply f^{-1} to the result. What do you get? Try the number 19.
     True or false? Since we always end up with the orginal input after a composition of this f and f^{-1} (both ways), that shows that f is symmetric across the line $y = x$. If you answer false, explain the correct composition that must be tested to establish symmetry across the line $y = x$.

   - Point on graph.

     Take any point on f, say $(31, 7)$. Reflect it across the line $y = x$. Show that the reflected point is on the graph of the inverse function, f^{-1}.
     True or false? Since the reflected point is on the graph of the inverse, it proves that f is symmetric across the line $y = x$. If answering false, explain the correct graph that the reflected point must be on to establish symmetry across the line $y = x$.

   - The easiest most basic method.
     State the simplest test for symmetry, once the equations for $f(x)$ and $f^{-1}(x)$ have been found, or once the graphs of f and f^{-1} have been found.

2. For each function, decide whether its graph is symmetric across the line $y = x$, and whether it has an inverse that is a function. If it does, find (if possible) a formula for the inverse.

   (a) $f(x) = 2$ (b) $f(x) = x$ (c) $f(x) = \dfrac{5x-3}{4}$ (d) $x^2 + y^2 = 0$

   (e) $f(x) = \dfrac{11x - x^2}{x}$ (f) $\dfrac{1}{x} + \dfrac{1}{y} = 1$ (g) $f(x) = x^2 - 4x + 7$, for $x \geq 2$ (domain $[2, \infty)$)

3. True or false? There is no function that is symmetric across both the y=axis and the line $y = x$. If false, find such a function.

4. True or false? It is possible to have a function that is symmetric through the origin and symmetric across the line $y = x$. If true, find an example.

ANSWERS FOLLOW

Answers.

1. The easiest way to find f^{-1} is to first write f as a verbal composition string, like this: subtract 3, then divide the result by 4.

   The inverse consists of reversed operations in reverse order: multiply by 4, then add 3. $f^{-1}(x) = 4x + 3$.

   Graphic: it is false. We need to see if the graph of f, by itself, is symmetric across the line $y = x$. The fallacy presented was that although the resulting graph picture will always be symmetric, it was because we added the inverse to the picture, and the inverse is really the same graph as f reflected (symmetrically placed) across the line $y = x$.

   Do not confuse the terms reflection and symmetry.

   The statement starting with "Every time" is true.

   Input-output model: false. It is always true, by defintion, that the composition of f and f^{-1} will return the original input as its output. To establish symmetry across the line $y = x$, we need to test the composition of f with itself, not with its inverse.

   Point on graph: false. The relection of any point on the graph across the line $y = x$ will always be on the graph of the inverse, by definition. The inverse is the result of that reflection, so it certainly contains every individual reflected point. For symmetry of f across the line $y = x$, we need to show that these reflected points are all on f, the original graph. In that case, every point of the graph is paired off with its symmetric partner and the graph is balanced about the line $y = x$.

   Most basic method: see if f and f^{-1} are equivalent. That would mean that they have the same solution set and that they have the same graph. In some cases it might be a little tricky to verify, but it is often the simplest method.

   Here: f and f^{-1} are both straight lines but they have different slopes, so they cannot have the same graph. Or take any old point on f, say $(27, 6)$ and show that it is not a solution to f^{-1}.

2. (a) Not symmetric; the inverse is not a function.
   (b) Symmetric; $f^{-1}(x) = x$, because the function is its own inverse.
   (c) Not symmetric; $f^{-1}(x) = \dfrac{4x+3}{5}$.
   (d) Symmetric over the real numbers; the only point on the graph is $(0, 0)$. That means that f and f^{-1} are both functions. Over the real numbers: $f(x) = f^{-1}(x) = 0$.
   (e) Not symmetric since the domain and range are not the same. $f^{-1}(x) = \dfrac{(11-x)^2}{11-x}$; that makes the domain and range work.
   (f) Symmetric. Use verbal strings to get the inverse to be: take reciprocal, then subtract 1, then change sign, then take reciprocal. It is tricky: this will translate back to $f^{-1}(x) = \dfrac{1}{1-\frac{1}{x}}$, but the obvious simplification to $f^{-1}(x) = \dfrac{x-1}{x}$ is incorrect, because it leads to the wrong domain.
   (g) Not symmetric. $f^{-1}(x) = 2 + \sqrt{x-3}$. First get it into standard form, then switch x and y and solve for y.

3. False. The only such function is the one with a single point $(0, 0)$ on its graph.

4. True. An example is $f(x) = 1/x$.