Logarithms and Logarithmic Functions

Math 130 *Kovitz*

Definition of the logarithm function

An exponent function is a one-to-one function of the form $f(x) = a^x$, where a is a positive number other than 1 and x is any real number. The base is a, the exponent is x, and a^x is the xth power of a. For example: with the base $a = 2$ and the exponent $x = 5$, 2^5 is called the 5th power of 2. The power is not the exponent; it is the result of raising to that exponent.

Its name is "the exponent function base a," abbreviated $f(x) = \exp_a x$.

The name of the function is $\exp_a$ and its formula is $y = a^x$.

The inverse of the function $\exp_a x$ will be called $\log_a x$, "the logarithm function base a." Because the exp function was one-to-one, the log will be a function.

Since the input to the exp function is the exponent and the output is the value of the base raised to that exponent, the log function will take a number and return an exponent. Key idea: *the logarithm is an exponent.*

The graph of the logarithm function

The graph of $\log_a x$ may be obtained by reflecting the exp function across the line $y = x$, reflecting some points of the exp function across the line $y = x$ and connecting them, or by switching the x and y in the original equation $y = a^x$ and graphing the new equation $x = a^y$.

Evaluating $\log_a p$, for p, a positive real number.

To find a formula for $f^{-1}(x)$ when $f(x) = \exp_a x$, first replace $f(x)$ with y, to get $y = a^x$.
Then switch x and y to have an equation of the inverse function: $x = a^y$.
Then try to solve for y. But in this case it is not possible to solve for the equation $x = a^y$ for y.
We are left with $y = \log_a x$, just the name of the new function, no formula.

By the implicit definition: $f^{-1}(x) = \log_a x$ is the unique value y such that $f(y) = \exp_a y = x$. That's $a^y = x$.
Verbally, $\log_a x$ is the exponent to which you must raise a to get x. In certain cases the value is known.
Simply stated $\log_a x$ asks: "to what exponent must we raise a to get x." So, for example, $\log_2 32 = 5$.

Some properties of a logarithmic Function and its graph

- The domain is the positive real numbers. $(0, \infty)$
- The range consists of all real numbers. $(-\infty, \infty)$
- The graph will have points in the first and fourth quadrants.
- The x-intercept is $(1, 0)$.
- There is no y-intercept.
- The graph is continuous because it is everywhere a connected curve.
- If $a > 1$: It is increasing and the graph rises from left to right. Also the negative y-axis is a vertical asymptote.
- If $0 < a < 1$: It is decreasing and the graph falls from left to right. Also the positive y-axis is a vertical asymptote.
- Any logarithmic function has the same shape as $y = \log_2 x$. Depending on the value of a, the base, it could be stretched or shrunk in the y-direction (when $a > 2$ it's shrunk making the graph narrower, and when $1 < a < 2$ it's stretched, making the graph more flat-shaped). If the base is less than 1, we know that the base $a = 1/c$, where $c > 1$. So, in addition the graph is a reflection of the graph of $y = \log_c x$ across the x-axis.

  Reason: it can be shown that $\log_{\frac{1}{c}} x = -\log_c x$. Call $\log_c x = p$. So $c^p = x$. Then verify that $\left(\frac{1}{c}\right)^{-p} = c^p$. This means that $\log_{\frac{1}{c}} x = -p = -\log_c x$. This shows that for any value of x the outputs (y's) of the two functions $\log_a x$ and $\log_c x$ have opposite signs.
- If the base is greater than 1, the graph is concave down; if the base is less than 1, the graph is concave up.
- The function is a one-to-one function since its inverse was a function.
- There is no maximum or minimum value of y, and the graph has no endpoint.
- The graph of a logarithmic function of the form $y = \log_a x$ has no symmetries.

Composition of logarithm and exponent functions with the same base

Since the composition of any function with its inverse will yield the identity function, giving $f^{-1}(f(a)) = a$ for all a in the domain of f and $f(f^{-1}(b)) = b$ for all b in the domain of f^{-1}, it follows that:

$$\log_a(\exp_a x) = x, \text{ meaning } \log_a(a^x) = x$$

and

$$\exp_a(\log_a x) = x, \text{ meaning } a^{\log_a x} = x.$$

Verbally: Find the exponent to which you must raise a to get a^x. It will of course be x.
Raise a to that exponent to which one must raise a to get x. What is the result? Of course it is x.

Writing equivalent equations

This involves the general relation of a function with its inverse: $f(a) = b$ implies $f^{-1}(b) = a$, and vice versa.

$p = \exp_a c$ $(p = a^c)$ has an equivalent logarithmic equation $\log_a p = c$.

$\log_a p = w$ has equivalent exponential equations $a^w = p$ and $\exp_a w = p$.

Keep these fomulas straight by remembering that the log is an exponent. So the numeric value of the log will be the exponent in the equivalent form. The number of which the log was taken becomes equal to that power of the base in the equivalent exponent form.

Rules of logarithms (Remember: logs are exponents and they will in some sense follow rules of exponents.)

The log of a product is the sum of the logs: $\log_a(c \cdot d) = \log_a c + \log_a d$, provided $c > 0$ and $d > 0$.

The log of a quotient is the difference of the logs: $\log_a(c \div d) = \log_a c - \log_a d$, provided $c > 0$ and $d > 0$.

The log of a power of a positive base is the product of the exponent and the log of the base: $\log_a(b^p) = p \cdot \log_a b$.

Be Careful: The log of the sum, $\log_a(b + c)$, has no easy simplification.

The product of the logs, $(\log_a c)(\log_a d)$, has no easy simplification.

A power of a log, $(\log_a c)^p$, has no easy simplification.

Do not confuse these with the actual rules of logarithms.

Special notation

Define the irrational number e as the result of finding $\left(1 + \dfrac{1}{n}\right)^n$, for n exceedingly large. It comes out to about $2.71828182845\ldots$.

Represent $\log_e$ by the notation ln, read "natural log." In calculus it will become apparent why the log to this base, rather than to the base 2 in the earlier examples, is the log to which logs to all other bases will be compared.

We have $\ln(e^a) = a$ and $e^{\ln c} = c$. Also $\ln a = c$ has an equivalent exponential form of $e^c = a$. This is easy to remember by noting that the log is an exponent. So c, being the log, is the exponent in the exponential form.

Other rules also carry over to ln: $\ln(c \cdot d) = \ln c + \ln d$, $\ln(c \div d) = \ln c - \ln d$, and $\ln(b^p) = p \cdot \ln b$.

Represent $\log_{10}$ by just log. When the base is unspecified, assume that it is 10. This is called the common log and it is most useful because numbers are generally represented in decimal form or in scientific notation.

Again $\log(10^a) = a$ and $10^{\log c} = c$. Also $\log a = c$ has an equivalent exponential form of $10^c = a$.

Other rules also carry over to log: $\log(c \cdot d) = \log c + \log d$, $\log(c \div d) = \log c - \log d$, and $\log(b^p) = p \cdot \log p$.

The change-of-base formula

$$\log_a x = \frac{\log_b x}{\log_b a}, \text{ for any valid base } b.$$

The log of any number to the old base is equal to the result of dividing the log of that number—taken to the new base—by the log of the old base—taken to the new base.

Special cases of the formula are $\log_a x = \dfrac{\log x}{\log a}$, where the base is ten, or $\log_a x = \dfrac{\ln x}{\ln a}$, where the base is e.

Log equations

In equations involving logarithms it is usually necessary to substitute each solution into the original equation to check for admissibility. If the equation is not valid over the real numbers the solution is extraneous and must be rejected. For example, a solution may lead to a logarithm of a negative number, which is not permissible.