Exponential Growth and Exponential Functions

Math 130 *Kovitz*

Definition 1 *A function is an exponential function if it can be expressed in the form*

$$f(t) = C_0 a^t.$$

In such a case f is said to exhibit exponential growth.

In such a case C_0 is called the initial value since $C_0 = f(0) = C_0 a^0 = C_0 \cdot 1$.

We call a the base or the growth factor per unit value of t. Every instance in which t increases by an amount h to become $t + h$ causes $f(t)$ to be multiplied by the factor

$$a^h \quad \text{(called the growth factor for time } h\text{)}$$

to become $f(t + h)$. In particular as t increases by 1 to become $t + 1$, $f(t)$ is multiplied by a to get $f(t + 1)$.

Proof:

$$f(t + h) = C_0 a^{t+h} = C_0 a^t \cdot a^h = f(t)a^h.$$

The function $f(t)$ can alternately be expressed in the form

$$f(t) = C_0 e^{rt},$$

where $e^r = a$ and $r = \ln a$.

In this form we call the value r the growth rate of the exponential function. Every exponential function has a single growth rate associated with it. The growth rate represents the instantaneous rate of increase, expressed as a fractional increase or decrease per unit time. This is not a universal defintion.

Definition 2 *The doubling time of an exponential function is the value D for which*

$$a^D = 2.$$

We get the derived formulae:

$$a = 2^{1/D}, \quad D = \ln 2/\ln a = \ln 2/r, \quad \text{and} \quad r = \ln 2/D.$$

Definition 3 *The half-life of an exponential function is the value H for which*

$$a^H = 1/2.$$

We get the derived formulae:

$$a = (1/2)^{1/H}, \quad H = -\ln 2/\ln a = -\ln 2/r, \quad \text{and} \quad r = -\ln 2/H.$$

Definition 4 *The semilog function is the function $h(t)$ where $h = \log \circ f$.*

Whenever f is an exponential function, its semilog function $\log \circ f$ is linear.

Proof:

$$(\log \circ f)(t) = \log(f(t)) = \log(C_0 a^t) = \log C_0 + \log(a^t) = \log C_0 + t \log a.$$

This linear function in such a case will have slope and y-intercept of $m = \log a$ and $b = \log C_0$. We also derive the equations $a = 10^m$ and $C_0 = 10^b$.

To get from a point (a, b) on the semilog graph to a corresponding point (c, d) on the original exponential function, we just note that since $a = \log c$, $c = \exp a = 10^a$, and that since $b = \log d$, $d = \exp b = 10^b$.

To summarize:

$$m = \log a, \quad b = \log C_0, \quad a = 10^m, \quad \text{and} \quad C_0 = 10^b.$$