

# Law of Cosine Problems

Math 130 Kovitz

1. Use the Law of Cosines to solve the triangle with sides of lengths 1 and 1, and an included angle of  $120^\circ$  between them. (Find the third side and the other two angles.) Use a shortcut to get the other two angles; do not use the Law of Cosines based on the three sides.

*Deductive reasoning question.*

Did you think, before applying any formula, that the third side must be greater than 1? Why? Also must the third side be less than 2? For what reason?

2. Use the Law of Cosines to solve the triangle with sides of lengths  $\sqrt{3}$  and 2, and an included angle of  $30^\circ$  between them. (Find the third side and the other two angles.)

Use a shortcut to get the other two angles; do not use the Law of Cosines based on the three sides.

Find the squares of the three sides. What do their values tell you about the triangle? about the side of length 2?

*Deductive reasoning question.*

Did you think, before applying any formula, that the third side must be less than 2? Why? Also must the third side be greater than  $2 - \sqrt{3}$ ? For what reason?

3. Use the Law of Cosines to solve the triangle with sides of lengths 1 and  $\sqrt{2}$ , and an included angle of  $45^\circ$  between them. (Find the third side and the other two angles.)

Use either of two possible shortcuts to get the other two angles; do not use the Law of Cosines based on the three sides.

The first shortcut is to list the three sides and decide what that tells us about the triangle.

The other shortcut is to find the squares of the three sides. What do their values tell you about the triangle? about the side of length  $\sqrt{2}$ ?

*Deductive reasoning question.*

Did you think, before applying any formula, that the third side must be less than  $\sqrt{2}$  and greater than  $\sqrt{2} - 1$ ? Why?

**Solutions follow below.**

# Solutions.

1.  $c^2 = a^2 + b^2 - 2ab \cos C$ .

$$c^2 = 1^2 + 1^2 - 2(1)(1) \cos 120^\circ = 2 - 2(-1/2) = 2 + 1 = 3.$$

That gives the third side as  $\sqrt{3}$ .

Think about it before proceeding. Is there any useful fact that will simplify the remaining work?

It is isosceles, so the other two angles must each be 30 degrees if the total is to be 180 degrees.

Or, if we missed that trick:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

The cosine of the angle between the sides 1 and  $\sqrt{3}$  is:

$$1^2 + (\sqrt{3})^2 - 1^2 / [2(1)(\sqrt{3})] = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}.$$

Take  $\cos^{-1}(\sqrt{3}/2) = 30^\circ$  to get that angle. No calculator needed for this one; it's a known angle.

The third angle is  $180^\circ - 120^\circ - 30^\circ = 30^\circ$ . Wasn't the shortcut better?

The largest angle is  $120^\circ$ , so the side opposite it will be the longest, letting us know before any work is done that it will turn out to be more than 1.

The sum of any two sides must be more than the third, so the third side must be less than 2.

2.  $c^2 = a^2 + b^2 - 2ab \cos C$ .

$$c^2 = (\sqrt{3})^2 + 2^2 - 2(\sqrt{3})(2) \cos 30^\circ = 7 - (4\sqrt{3})(\sqrt{3}/2) = 7 - 6 = 1.$$

That gives the third side as 1.

Think about it before proceeding. Is there any useful fact that will simplify the remaining work?

The squares of the sides are 1, 3, and 4. That means  $a^2 + b^2 = c^2$  applies and the triangle is a right triangle with the hypotenuse equal to the longest side: 2.

The angles are  $30^\circ$ ,  $90^\circ$ , and  $180^\circ - 30^\circ - 90^\circ = 60^\circ$ .

Or, if we missed that trick:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

The cosine of the angle between the sides 1 and 2 is:

$$1^2 + 2^2 - (\sqrt{3})^2 / [2(1)(2)] = \frac{2}{4} = 1/2.$$

Take  $\cos^{-1}(1/2) = 60^\circ$  to get that angle. No calculator needed for this one; it's a known angle.

The third angle is  $180^\circ - 30^\circ - 60^\circ = 90^\circ$ . Wasn't the shortcut better?

The largest angle cannot be  $30^\circ$ , so the side opposite it will not be the longest, letting us know before any work is done that it will turn out to be less than 2. So 2 must be the longest side and the angle opposite it,  $90^\circ$  in this case, must be the largest angle.

The sum of any two sides must be more than the third, so unless the third side is more than  $2 - \sqrt{3}$ , there is no triangle.

3.  $c^2 = a^2 + b^2 - 2ab \cos C$ .

$$c^2 = 1^2 + (\sqrt{2})^2 - 2(1)(\sqrt{2}) \cos 45^\circ = 3 - (2\sqrt{2})(\sqrt{2}/2) = 3 - 2 = 1.$$

That gives the third side as 1.

Think about it before proceeding. Is there any useful fact that will simplify the remaining work?

The squares of the sides are 1, 1, and 2. That means  $a^2 + b^2 = c^2$  applies and the triangle is a right triangle with the hypotenuse equal to the longest side:  $\sqrt{2}$ .

The angles are  $45^\circ$ ,  $90^\circ$ , and  $180^\circ - 45^\circ - 90^\circ = 45^\circ$ .

Or since it is isosceles, the angle opposite the other side of length 1 must also be  $45^\circ$ , leaving a third angle of  $90^\circ$ .

Or, if we missed that trick:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

The cosine of the angle between the sides 1 and 1 is:

$$1^2 + 1^2 - (\sqrt{2})^2 / [2(1)(1)] = \frac{0}{2} = 0.$$

Take  $\cos^{-1}(0) = 90^\circ$  to get that angle. No calculator needed for this one; it's a known angle.

The third angle is  $180^\circ - 45^\circ - 90^\circ = 45^\circ$ . Wasn't the shortcut better?

The largest angle cannot be  $45^\circ$ , so the side opposite it will not be the longest, letting us know before any work is done that it will turn out to be less than  $\sqrt{2}$ . So  $\sqrt{2}$  must be the longest side and the angle opposite it,  $90^\circ$  in this case, must be the largest angle.

The sum of any two sides must be more than the third, so unless the third side is more than  $\sqrt{2} - 1$ , there is no triangle.

Any way this is done, it is an isosceles right triangle.