Answers to the Sample of Typical Final Examination Problems

Math 130 Precalculus for the December ??, 2019 Final Exam

1. $\dfrac{f(2+h)-f(2)}{h} = \dfrac{\frac{1}{3+h}-\frac{1}{3}}{h} = \dfrac{\frac{-h}{3(3+h)}}{h} = \dfrac{-1}{3(3+h)}$

2. (a) f is odd because it simplifies to $\frac{3x^2-1}{x(x^2-1)}$, and when x has opposite sign, so does this fraction.
   (b) g is neither even nor odd because $g(3)=4$ while $g(-3)=2$.
   (c) h is odd because $h(a)=|a+3|-|a-3|$ and $h(-a)=|-a+3|-|-a-3|=|a-3|-|a+3|=-h(a)$. For support, show that $h(4)=6$ and $h(-4)=-6$.
   (d) i is even because $i(x)=\sqrt{16-x^2}$ showing that it is even by the rule of even powers and also that the domain is balanced, being $[-4,4]$.
   (e) j is neither even nor odd because the domain, $x \geq 1$, is not balanced.

3. (a) False: $(f \circ g)(x) = |x|$ for all real numbers.
   (b) f and g are not inverse functions. The domains and ranges do not match up properly.

4. $f^{-1}(x) = \dfrac{3x-4}{5x}$. It was found by solving $x = \dfrac{4}{-5y+3}$ for y. It is a function.

   As a verbal string it is divide by 4; take reciprocal; subtract 3; change sign; divide by 5. This is the formula
$$-\frac{1}{5}\left(\frac{4}{x}-3\right).$$

   The two answers, while different formulas, are algebraically equivalent.

   The orginal verbal string was (before inversion) multiply by 5, change sign, add 3, take reciprocal, multiply by 4. All that had to be done was to find the inverse of each operation and apply them in the opposite order as they were in the original function.

   Because this problem is not easily conceptualized, one should apply the two results to a number of his choosing to see if they work.
   Let $x=3$, so $f(3) = \frac{4}{-15+3} = \frac{4}{-12} = -\frac{1}{3}$.

   Then $f^{-1}(-1/3) = \frac{3(-1/3)-4}{5(-1/3)} = \frac{-5}{-5/3} = 3$. It checks.

   And the verbal: $-1/3$ divided by 4 is $-1/12$, reciprocal is -12, subtract 3 gives -15, change sign gives 15, divide by 5 gives 3, as expected.

   $f(1)=-2$, so $a=-2$.
   $f^{-1}(-2) = \frac{3(-2)-4}{5(-2)} = \frac{-10}{-10} = 1$.

   It supports the formula for f^1 because $f^{-1}(f(1)) = 1$ as the rule for inverses requires.

5. Let $x=$ the width and let $z=$ the length. The area $= xz$.
   Since $x+z=26$, we have $z=26-x$. Substituting $26-x$ for z, we find that the area $= xz = x(26-x) = 26x-x^2 = -x^2+26x$. So for the area we have $f(x)=-x^2+26x$ with $a=-1<0$. That means that the parbola opens down and has a maximum value of $f(x)$.
$$k = c - \frac{b^2}{4a} = 0 - \frac{26^2}{4(-1)} = -\frac{576}{-4} = \frac{-576}{-4} = 169 \text{ square feet}$$

   The maximum area is equal to the maximum value of $f(x)$, which equals the maximum value of y, which is called k. Remember that the quantity to be maximized is represented by y.
   This problem could also be done by completing the square the long way.
   We find that $f(x) = -x^2+26x = -(x^2-26x) = -(x^2-26x+169-169) = -(x-13)^2+169$.
   Thus $f(x) = -(x-13)^2+169$ and the vertex is $(13,169) = (h,k)$.

6. Originally $y = \log_2 x$.
   Now $y = \log_2(16x^2) = \log_2 16 + \log_2 x^2 = 4 + 2\log_2 x$. The y ends up 4 more than double the previous y.

7. (a) 1/2 (b) 1/2

8. (a) True.
   $\log_a uv = \log_a u + \log_a v$
   This is the first property of logarithms.
   (b) False.
   $(\log_a u) \div (\log_a v) = \log_a(u - v)$
   $2 = \log 100 \div \log 10 \neq \log(100 - 10) = \log 90 \approx 1.95$
   (c) True.
   $(\log_a b)(\log_b a) = (\log_a b)\left(\frac{\log_a a}{\log_a b}\right) = \log_a a = 1$. The key was changing the base of $\log_b a$ to base a.

9. (a) 2.838 (b) −0.8095 (c) 7.6485 (d) 0.87 (e) 4.7455 (f) −6.392 (g) 4.5495 (h) 0.69933

10. (a) $x = 20$ (b) $x = 500/33$ or $x = 15\frac{5}{33}$ (c) $x = 25$

11. A valid solution must greater than 1/2.

    $\log_2\left(\frac{x}{2x-1}\right) = -3.$

    $2^{-3} = \frac{x}{2x-1}.$

    $\frac{1}{8} = \frac{x}{2x-1}.$

    $8x = 2x - 1$, and $x = -\frac{1}{6}$. But this proposed solution is not greater than 1/2.

    Conclusion: there is no solution to this equation.

12. Any solution must be a positive number for which $1 - 3x > 0$ also. That means $x < 1/3$, and we have $0 < x < 1/3$. A valid solution must lie in the interval $(0, 1/3)$.

    By the product rule for logarithms: $\log_2(x - 3x^2) = -4$.
    Rewriting in exponent form: $2^{-4} = x - 3x^2$.
    Then $3x^2 - x + \frac{1}{16} = 0$.
    This could be solved by the quadratic formula, or by multiplying out by 16 then factoring, or by straight factoring using fractions.

    Quadratic formula: $\dfrac{1 \pm \sqrt{1 - 4(3)(1/16)}}{6} = \dfrac{1 \pm \frac{1}{2}}{6}.$
    The answers are $\frac{3/2}{6} = \frac{3}{12} = 1/4$ and $\frac{1/2}{6} = 1/12$.

    Multiply out: $48x^2 - 16x + 1 = 0$, factor as $(12x - 1)(4x - 1) = 0$, so $12x - 1 = 0$ and $x = 1/12$,
    or $4x - 1 = 0$ and $x = 1/4$.

    Simple factoring with fractions: $(3x - 1/4)(x - 1/4) = 0$ will give $3x - 1/4 = 0$, $x = 1/12$,
    and $x - 1/4 = 0$, $x = 1/4$.
    Check when $x = 1/4$:
    $\log_2(1/4) + \log_2(1 - 3(1/4)) = -2 + \log_2(1/4) = -2 + (-2) = -4$; it's OK.

13. Any solution must be a positive number for which $1 - 2x > 0$ also. That means $x < 1/2$, and we have $0 < x < 1/2$. A valid solution must lie in the interval $(0, 1/2)$.

    By the product rule for logarithms: $\log_2(x - 2x^2) = -3$.
    Rewriting in exponent form: $2^{-3} = x - 2x^2$.
    Then $2x^2 - x + \frac{1}{8} = 0$.
    This could be solved by the quadratic formula, or by multiplying out by 8 then factoring, or by straight factoring using fractions.

    Quadratic formula: $\dfrac{1 \pm \sqrt{1 - 4(2)(1/8)}}{4} = \dfrac{1 \pm 0}{4} = 1/4.$

Multiply out: $16x^2 - 8x + 1 = 0$, perfect square $(4x-1)^2 = 0$, so $4x - 1 = 0$ and $x = 1/4$.

Simple factoring with fractions: $(2x - 1/2)(x - 1/4) = 0$ will give $2x - 1/2 = 0$, $x = 1/4$,
and $x - 1/4 = 0$, $x = 1/4$.

Check:

$\log_2(1/4) + \log_2(1 - 2(1/4)) = -2 + \log_2(1/2) = -2 + (-1) = -3$; it's OK.

14. (a) $\dfrac{3+\sqrt{5}}{2}$ (b) 9/8 or 1.125

15. (a) $\frac{3}{8}\pi$ or $67.5°$.
 (b) $157.5°$
 (c) $\frac{11\pi}{15}$.
 (d) 125 feet, because $150° = \frac{5\pi}{6}$ and $\frac{150}{\pi}\left(\frac{5\pi}{6}\right)$ feet $= 125$ feet after cancellation.

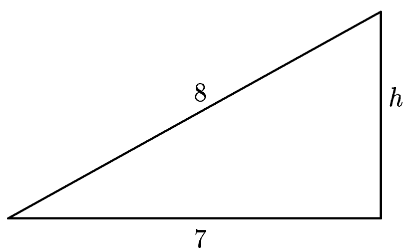
16. Convert $80°$ to radians: $80° \times \frac{\pi \text{ radians}}{180°} = \frac{4\pi}{9}$.

 80 feet $= \frac{4\pi}{9} \times r$. Since $\frac{4\pi}{9}$ is a unitless ratio, do not include the word radians in this equation.

 $r = \frac{180}{\pi}$ feet, after solving.

17. $\sin\theta = \dfrac{\sqrt{15}}{8}$ $\cos\theta = \dfrac{7}{8}$ $\tan\theta = \dfrac{\sqrt{15}}{7}$ $\csc\theta = \dfrac{8\sqrt{15}}{15}$ $\cot\theta = \dfrac{7\sqrt{15}}{15}$

 $\sin 2\theta = \frac{7}{32}\sqrt{15}$ $\cos 2\theta = 17/32$



 To find the third side, use $h^2 + 7^2 = 8^2$ so $h = \sqrt{64-49} = \sqrt{15}$.
 Then $\sec(90° - \theta) = 1/\cos(90° - \theta) = 1/\sin\theta = \csc\theta = \frac{8\sqrt{15}}{15}$.
 And $\csc^2\theta - 1 = \cot^2\theta = \left(\frac{7}{\sqrt{15}}\right)^2 = 49/15$.

18. (a) True.
 $\csc x - \sin x = \frac{1-\sin^2 x}{\sin x} = \frac{\cos^2 x}{\sin x} = \left(\frac{\cos x}{\sin x}\right)\cos x$.
 (b) False.
 (c) False.

19. (a) $\cos x(1+\tan x)(1-\tan x) = \cos x(1-\tan^2 x) = \cos x(1-(\sec^2 x - 1)) = \cos x(2-\sec^2 x) =$
 $$2\cos x - \frac{1}{\cos x} = \frac{2\cos^2 x - 1}{\cos x}.$$ This answer could also be given as $\dfrac{\cos 2x}{\cos x}$.
 (b) $\tan x\cos^2 x = (\sin x/\cos x)\cos^2 x = \sin x\cos x = \frac{1}{2}\sin 2x$.
 (c) $\cos^4 x - \sin^4 x = (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) = (1)(\cos 2x) = \cos 2x$
 (d) $\frac{1+\cot^2 x}{\sin x} = \frac{\csc^2 x}{\sin x} = \csc^2 x\left(\frac{1}{\sin x}\right) = \csc^2 x(\csc x) = \csc^3 x$.
 (e) $\frac{\sec x}{\csc x} = \left(\frac{1}{\cos x}\right)\left(\frac{1}{\csc x}\right) = \left(\frac{1}{\cos x}\right)\sin x = \frac{\sin x}{\cos x} = \tan x$.
 (f) $\frac{\sec x}{\sin x} = \sec x\left(\frac{1}{\sin x}\right) = \left(\frac{1}{\cos x}\right)\left(\frac{1}{\sin x}\right) = \frac{1}{\sin x\cos x} = \frac{1}{(1/2)\sin 2x} = 2\csc 2x$.

20. $41.99°$ and $180° - 41.99° = 138.01°$. For the sine, the first solution is obtained from the arcsine and a second solution is $180°$ minus the first.

21. $105.07°$ and $360° - 105.07° = 254.93°$.

22. (a)

$$2\sin x\cos x = -\cos x$$
$$2\sin x\cos x + \cos x = 0$$
$$\cos x(2\sin x + 1) = 0$$

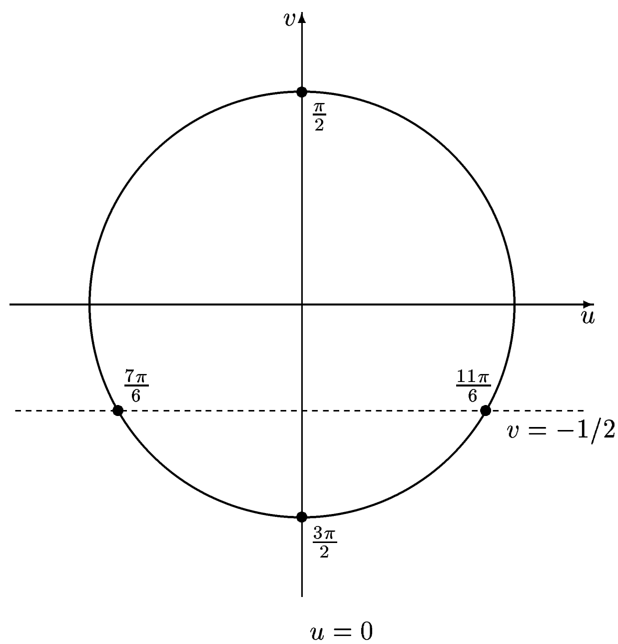
$\cos x = 0$ or $2\sin x + 1 = 0$

$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$

$2\sin x = -1$

$\sin x = -1/2$

$x = \frac{7\pi}{6}$ or $x = \frac{11\pi}{6}$



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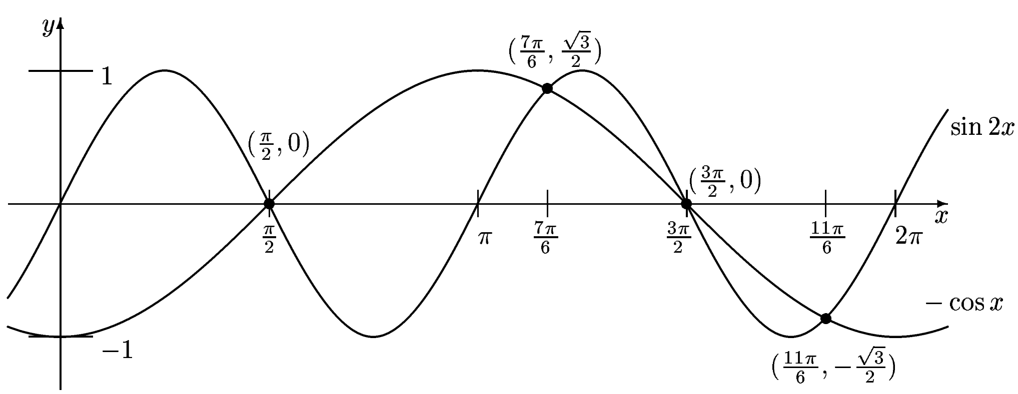


23. (a)

$$\begin{aligned}2\sin x\cos x &= -\sin x\\ 2\sin x\cos x+\sin x &= 0\\ \sin x(2\cos x+1) &= 0\end{aligned}$$

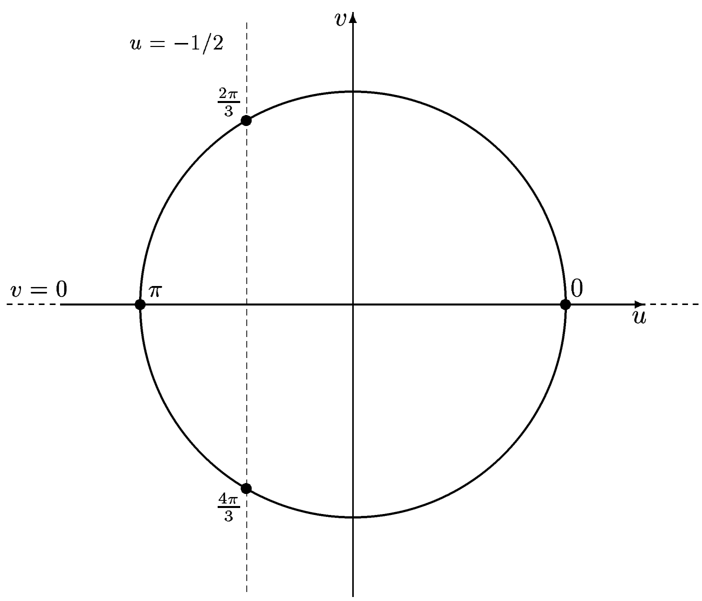
$\sin x = 0$ or $2\cos x + 1 = 0$

$x = 0$ or $x = \pi$ or $x = 2\pi$

$2\cos x = -1$

$\cos x = -1/2$

$x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$



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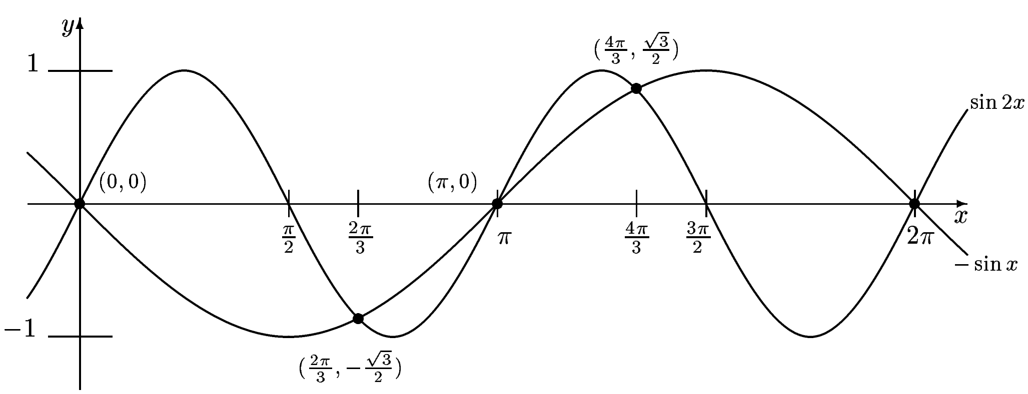


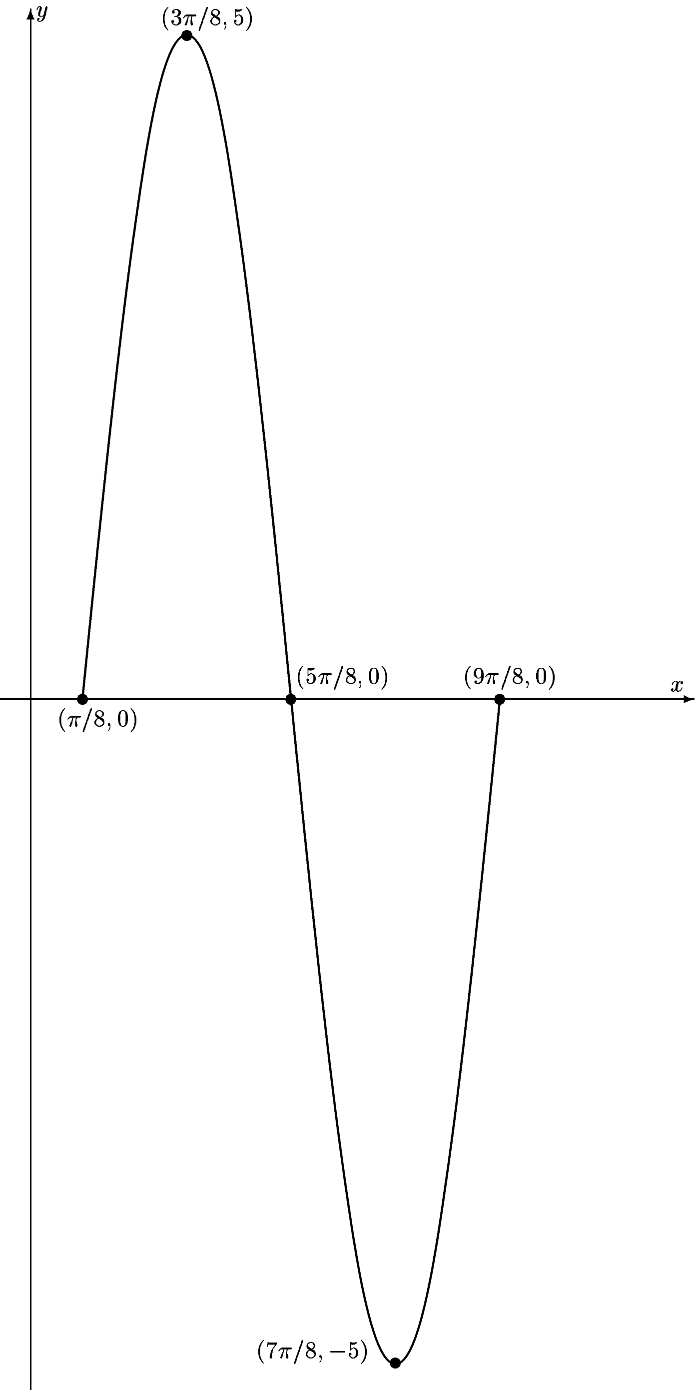
24.

The period is $\dfrac{2\pi}{|a|} = \dfrac{2\pi}{2} = \pi$.

The amplitude is 5.

Factor as $\quad 5\sin\left[2\left(x - \frac{\pi}{8}\right)\right]$.

So shift the graph of $y = 5\sin 2x$ $\pi/8$ to the right.



25. $\sqrt{1 - \dfrac{1}{x^2}}$ or $\dfrac{\sqrt{x^2-1}}{|x|}$

A common wrong answer is $\dfrac{\sqrt{x^2-1}}{x}$.

It is a major error to draw a right triangle with hypotenuse x and an opposite leg 1 to represent this formula. A hypotenuse must always be positive and assigning it a value of x allows that x could be negative, leading to an inadmissible situation.

A correct assignment to the hypotenuse is 1, with the opposite leg being assigned $1/x$.