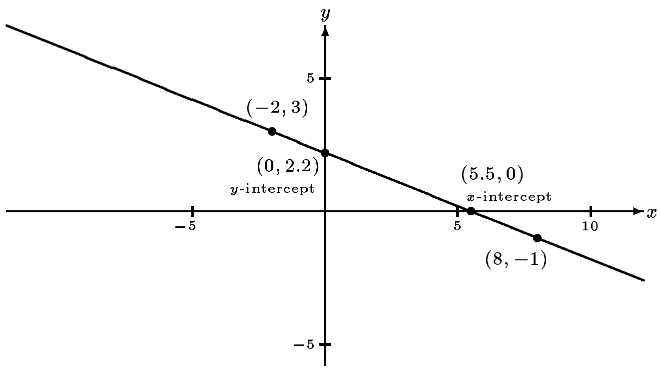
Answers to the Sample Final Examination Problems

Math 130 Precalculus of the December ??, 2019 Final Exam

1. $$y = -\tfrac{2}{5}x + \tfrac{11}{5} \quad \text{or} \quad y = -0.4x + 2.2$$

Also $y - 3 = -\frac{2}{5}(x+2)$ or $y + 1 = -\frac{2}{5}(x-8)$

Also $2x + 5y = 11.$



Work: $m = \frac{-1-3}{8-(-2)} = \frac{-4}{10} = -2/5$ and $y - 3 = -\frac{2}{5}(x+2).$

2. The center is at the midpoint of the diameter: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = (1/2, 2).$

   The distance between the given points is $\sqrt{(-2-3)^2 + (-4-8)^2} = \sqrt{25+144} = \sqrt{169} = 13.$

   That was the length of the diameter; the radius has length $13/2 = 6.5.$

   $\left(x - \frac{1}{2}\right)^2 + (y-2)^2 = 6.5^2.$

3. $\dfrac{f(2+h) - f(2)}{h} = \dfrac{(8 + 8h + 2h^2 + 6 + 3h - 1) - (2(2^2) + 3(2) - 1)}{h} = \dfrac{11h + 2h^2}{h} = 11 + 2h,\ h \neq 0.$

4. (a) The graph of $y = \sqrt{x}$ was moved 5 units to the left and 5 units up.

   (b) The graph of $y = \sqrt{x}$ was stretched vertically by a factor of 2 and moved 3 units to the right.

   (c) It is best to rewrite this as $i(x) = -\sqrt{-(x+3)}$.
   Then graph an intermediate function, $y = \sqrt{-x}$, first. Its graph is the graph of $f(x)$ reflected across the y-axis.
   Then replace x with $(x+3)$ to get $y = \sqrt{-(x+3)}$. This moves the previous graph 3 units to the left.
   Finally, $i(x) = -\sqrt{-x-3}$ is just the previous graph reflected across the x-axis, since all y values will assume opposite sign.

   Answer: The graph of $y = \sqrt{x}$ was reflected across the y-axis, then shifted 3 units to the left, and then reflected across the x-axis.

   Verify by noting that the point $(9,3)$ on f will end up as the point $(-12,-3)$ on i and that the point $(-12,-3)$ is indeed a solution of the equation for i because $-3 = -\sqrt{-(-12)-3}$.

5. (a) $f \circ g(x) = f(g(x)) = \sqrt{(x^2+4)-4} = \sqrt{x^2} = |x|$ and $g \circ f(x) = g(f(x)) = (\sqrt{x-4})^2 + 4 = x - 4 + 4 = x.$
   The domain of $f \circ g$ is all reals since g always has an output of 4 or more which, when input into f guarantees that the number under the square root will be positive.

   The domain of $g \circ f$ is $[4, \infty)$, as that is the domain of f.

   (b) f: The domain is $[4, \infty)$ and the range is $[0, \infty)$.

   g: The domain is all reals and the range if $[4, \infty)$.

   Since the domain of g is not the same as the range of f, f and g are not inverse functions.

6. (a) The range is $[0, \infty]$.

   The x-intercept is at $(-1, 0)$ and the y-intercept is at $(0, 1)$.

   (b) Yes, it is steadily increasing.

   To find the inverse: write $y = \sqrt{x+1}$, switch x and y to get: $x = \sqrt{y+1}$, then solve for y.

   $x^2 = y + 1$, so $y = x^2 - 1$. The formula for the inverse of f is: $f^{-1}(x) = x^2 - 1$.

   However, the domain of the inverse must be the same as the range of f. The inverse is $f^{-1}(x) = x^2 - 1$ for $x \geq 0$ only.

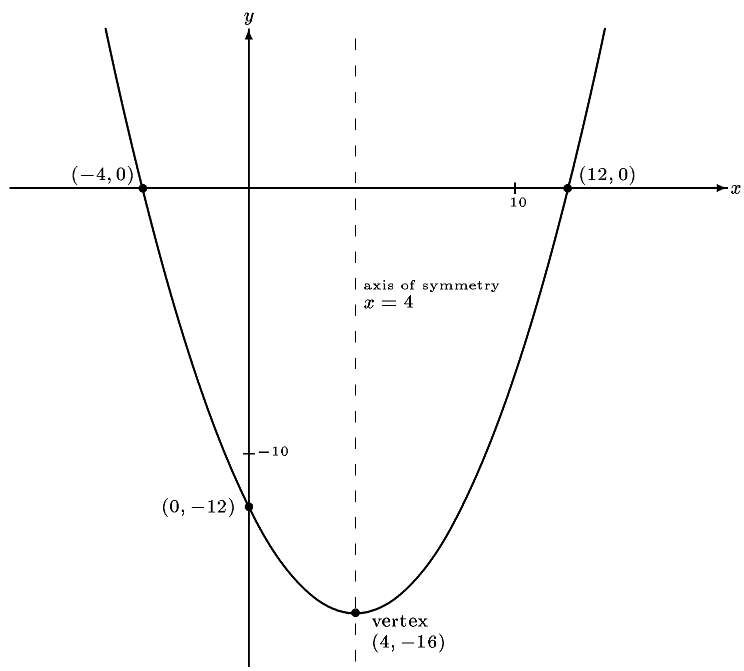
   The y-intercept is at $(0, -1)$ and the x-intercept is at $(1, 0)$. Just switch the x- and y-coordinates of the previous points, because the inverse is the reflection across the line $y = x$.

7. $y = \frac{1}{4}(x-4)^2 - 16$.

The steps: $y = \frac{1}{4}(x^2 - 8x) - 12$, $y = \frac{1}{4}(x^2 - 8x + 16 - 16) - 12$, $y = \frac{1}{4}[(x-4)^2 - 16] - 12$.

Be very careful to distribute to the dangling term, -16. The $\frac{1}{4}(-16)$ becomes -4.

$$\text{So, the standard form is } y = \frac{1}{4}(x-4)^2 - 16.$$



8. Call the two numbers w and x. It is best to avoid using the variable y because of the potential confusion with the actual output.

   The product is $p = wx$. This problem can be reduced to two variables by using the conditions relating w and x linearly.

   $\left(\frac{1}{4}\right) w + x = 5$.

   Solve for w to get $w = 20 - 4x$.

   Substitute in the original equation to get $p = (20 - 4x)x = 20x - 4x^2$.

   $p = -4x^2 + 20x$ is a quadratic function in x. It opens down and has a maximum.

   The maximum product—the maximum value of this function—is 25, obtained from the vertex formula. It occurs when the 2nd number $x = 2.5$, because by the vertex formula $h = -b/2a = -20/(-8) = 2.5$.

   From this result, let us construct the original ordered pair that has the maximum product.

   The product is 25, the second number is 2.5, and the first number is $20 - 4(2.5) = 10$, so the ordered pair was $(10, 2.5)$. The product checks as $10 \times 2.5 = 25$.

9. All are true.

10. $\log_b \frac{1}{2} = \log_b 1 - \log_b 2 = 0 - 3/4 = -3/4$.

    To evaluate $\log_2 b$, change the base to b and get $\log_2 b = \frac{\log_b b}{\log_b 2} = \frac{1}{3/4} = 4/3$.

    $b^{3/4} = 2$, so raise each side to the 4/3 power to get $b = 2^{4/3}$. It rounds off to about 2.52.

11. (a) True.
    (b) False.
    (c) True. Be careful. The denominator is the log of the square root, not the square root of the log.
    (d) False. It is not true when a is negative. But change the right side to $2\log|a| - 2$ and it is true.
    (e) True.
    (f) False.

12. (a) False.
    (b) False.
    (c) True.
    (d) False.
    (e) True.

13. (a) True.
    (b) True: it is $\log_2 5$, which equals (upon change to base 10) $\dfrac{\log 5}{\log 2}$.

14. (a) $\log_a 32 + \log_a \frac{1}{4} = \log_a(32 \times \frac{1}{4}) = \log_a 8 = \log_a 2^3 = 3\log_a 2 = 3 \times 0.2 = 0.6$.
    (b) $\log_a 4\sqrt{2} = \log_a 4 + \log_a \sqrt{2} = \log_a 2^2 + \log_a 2^{1/2} = 2\log_a 2 + (1/2)\log_a 2 = 0.4 + 0.1 = 0.5$.
    (c) $\frac{\log_a 5}{\log_a 2} = \log_2 5$, by change of base.
    Then multiply out to get $\log_a 5 = (\log_a 2)(\log_2 5) = 0.2(\frac{\log 5}{\log 2}) = 0.2(0.69897/0.30103) \approx 0.4644$.

15. Only solutions where $x > 1$ will be valid.

    $\log\left(\frac{1}{x-1}\right) = 2$.
    $10^2 = \frac{x}{x-1}$.
    $100 = \frac{x}{x-1}$.
    $100x - 100 = x$, $99x = 100$, $x = 100/99$. That's OK; it is greater than 1.

16. (a) 7.5 hours.
    (b) False. There were exactly 8 pounds of frut flies at time 17 hours, not at time 8 hours.
    (c) $2^{1.8} \approx 3.4822$ pounds of fruit flies.
    (d) Exactly at time 39 hours and 30 minutes.

17. (a) $\sin\theta = 1/\csc\theta = \frac{1}{5/2} = 2/5$.
    (b) $1 + \cot 2\theta = \csc^2\theta$, so $\cot^2\theta = (5/2)^2 - 1 = 21/4$ and $\cot\theta = \sqrt{21}/2$.
    (c) $\cos 2\theta = 1 - \sin^2 2\theta = 1 - 2(2/5)^2 = 1 - 2(4/25) = 1 - 8/25 = 17/25 = 0.68$.
    (d) $\sin(30° + \theta) = \sin 30° \cos\theta + \cos 30° \sin\theta = (1/2)(\sqrt{21}/5) + (\sqrt{3}/2)(2/5) = \sqrt{21}/10 + (2\sqrt{3})/(2 \times 5) = \sqrt{21}/10 + \sqrt{3}/5 \approx 0.8046677$.

    This problem could be checked on a calculator by getting $\sin^{-1} 0.4 \approx 23.5782°$.
    For part (c): Double 23.5782° to get 47.1564°. Its cosine is 0.68000, to 5 decimals.
    For part (d): add 30° to 23.5782° and take the sine of the result. The sine of 53.5782° is close to 0.804668.

18. $\cos 30° = \sqrt{3}/2 = 4/h$. So $h = 8/\sqrt{3} = \frac{8}{3}\sqrt{3}$ after rationalization.

19. $\pi/6$ and $2\pi - \pi/6$. So $\pi/6$ and $11/\pi/6$. For the cosine, the primary solution is always the arccosine and the second one is either 2π minus the first solution or minus the first solution.

20. $\sin 165° = \sin 120° \cos 45° + \cos 120° \sin 45° = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{6}/4 - \sqrt{2}/4$.

21. $x = 400$.

    Solution: $\log\left(\frac{25x}{4(5-\sqrt{x})}\right) = 2$.

    $\frac{25x}{5+\sqrt{x}} = 100$.

    Divide both sides by 25 to get: $\frac{x}{4(5+\sqrt{x})} = 4$.

    $x = 16(5 + \sqrt{x})$.

    $x - 16\sqrt{x} - 80 = 0$.

    Let $a = \sqrt{x}$. Then: $a^2 - 16a - 80 = 0$ and $(a - 20)(a + 4) = 0$.

    a cannot be negative, so -4 is rejected and $a = 20$.

    $x = a^2 = 400$.

    Check: $\log\left(\frac{10{,}000}{4}\right) - \log(5 + \sqrt{400}) = 2$.

    $\log 2500 - \log(25) = \log(2500/25) = \log 100 = 2$, so it checks.

22. (a) i. $x = \arctan\left(\dfrac{1}{2}\right)$

    ii. $\approx 26.56505°$

    (b) $\csc 2x = \csc 53.1301° = 1/\sin 53.1301° = 1/0.8 = 1.25$.

    and

    $1 + \tan^2 26.56505 = 1 + .5^2 = 1.25$. Verified. (Slight rounding off was necessary.)