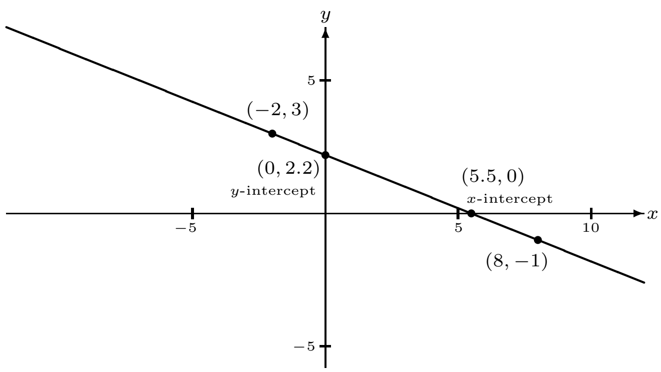
Answers to the Sample of Typical Final Examination Problems

Math 130 Precalculus for the December 22, 2017 Final Exam

1. $$y = -\tfrac{2}{5}x + \tfrac{11}{5} \quad \text{or} \quad y = -0.4x + 2.2$$

Also $y - 3 = -\frac{2}{5}(x+2)$ or $y + 1 = -\frac{2}{5}(x-8)$

Also $2x + 5y = 11.$



Work: $m = \frac{-1-3}{8-(-2)} = \frac{-4}{10} = -2/5$ and $y - 3 = -\frac{2}{5}(x+2)$.

2. $(x-11)^2 + (y-10.5)^2 = 7.5$. The center is at $(11, 10.5)$ and the radius is 7.5. The highest point is one radius length above the center. It is at $(11, 10.5+7.5) = (11, 18)$.

First find the midpoint of the diameter: $(\frac{5+17}{2}, \frac{6+15}{2})$. Ths will be the center.

Then find the distance between the endpoints of the diameter:
$\sqrt{(17-5)^2 + (15-9)^2} = \sqrt{144+81} = \sqrt{225} = 15.$

The radius will be half of that, so 7.5 is the radius.

3. $\dfrac{f(2+h) - f(2)}{h} = \dfrac{(8 + 8h + 2h^2 + 6 + 3h - 1) - (2(2^2) + 3(2) - 1)}{h} = \dfrac{11h + 2h^2}{h} = 11 + 2h,\ h \neq 0.$

4. (a) f is odd because it simplifies to $\frac{3x^2-1}{x(x^2-1)}$, and when x has opposite sign, so does this fraction.
   (b) g is neither even nor odd because $g(3) = 4$ while $g(-3) = 2$.
   (c) h is odd because $h(a) = |a+3| - |a-3|$ and $h(-a) = |-a+3| - |-a-3| = |a-3| - |a+3| = -h(a)$. For support, show that $h(4) = 6$ and $h(-4) = -6$.
   (d) i is even because $i(x) = \sqrt{16 - x^2}$ showing that it is even by the rule of even powers and also that the domain is balanced, being $[-4, 4]$.
   (e) j is neither even nor odd because the domain, $x \geq 1$, is not balanced.

5. (a) False: $(f \circ g)(x) = |x|$ for all real numbers.
   (b) f and g are not inverse functions. The domains and ranges do not match up properly.

6. (a) The range is $[0, \infty]$.

   The x-intercept is at $(-1, 0)$ and the y-intercept is at $(0, 1)$.

   (b) Yes, it is steadily increasing.

   To find the inverse: write $y = \sqrt{x+1}$, switch x and y to get: $x = \sqrt{y+1}$, then solve for y.

   $x^2 = y + 1$, so $y = x^2 - 1$. The formula for the inverse of f is: $f^{-1}(x) = x^2 - 1$.

   However, the domain of the inverse must be the same as the range of f. The inverse is $f^{-1}(x) = x^2 - 1$ for $x \geq 0$ only.

   The y-intercept is at $(0, -1)$ and the x-intercept is at $(1, 0)$. Just switch the x- and y-coordinates of the previous points, because the inverse is the reflection across the line $y = x$.

7. $f^{-1}(x) = \dfrac{3x-4}{5x}$. It was found by solving $x = \dfrac{4}{-5y+3}$ for y. It is a function.

As a verbal string it is divide by 4; take reciprocal; subtract 3; change sign; divide by 5. This is the formula

$$-\frac{1}{5}\left(\frac{4}{x}-3\right).$$

The two answers, while different formulas, are algebraically equivalent.

The orginal verbal string was (before inversion) multiply by 5, change sign, add 3, take reciprocal, multiply by 4. All that had to be done was to find the inverse of each operation and apply them in the opposite order as they were in the original function.

Because this problem is not easily conceptualized, one should apply the two results to a number of his choosing to see if they work.
Let $x = 3$, so $f(3) = \frac{4}{-15+3} = \frac{4}{-12} = -\frac{1}{3}$.

Then $f^{-1}(-1/3) = \frac{3(-1/3)-4}{5(-1/3)} = \frac{-5}{-5/3} = 3$. It checks.

And the verbal: $-1/3$ divided by 4 is $-1/12$, reciprocal is -12, subtract 3 gives -15, change sign gives 15, divide by 5 gives 3, as expected.

$f(1) = -2$, so $a = -2$.
$f^{-1}(-2) = \frac{3(-2)-4}{5(-2)} = \frac{-10}{-10} = 1$.

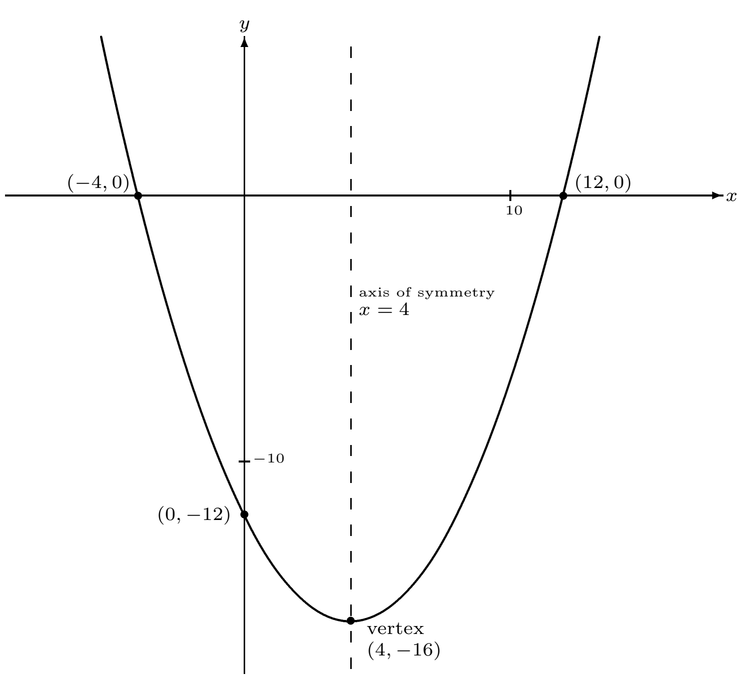
It supports the formula for f^1 because $f^{-1}(f(1)) = 1$ as the rule for inverses requires.

8. $y = \frac{1}{4}(x-4)^2 - 16.$

The steps: $y = \frac{1}{4}(x^2-8x) - 12$, $y = \frac{1}{4}(x^2-8x+16-16) - 12$, $y = \frac{1}{4}[(x-4)^2-16] - 12$.

Be very careful to distribute to the dangling term, -16. The $\frac{1}{4}(-16)$ becomes -4.

So, the standard form is $y = \dfrac{1}{4}(x-4)^2 - 16$.



9. Originally $y = \log_2 x$.

Now $y = \log_2(32x) = \log_2 32 + \log_2 x = 5 + \log_2 x$. The y ends up 5 larger, so the change in y is 5.

10. $\log_b \frac{1}{2} = \log_b 1 - \log_b 2 = 0 - 3/4 = -3/4$.

    To evaluate $\log_2 b$, change the base to b and get $\log_2 b = \frac{\log_b b}{\log_b 2} = \frac{1}{3/4} = 4/3$.

    $b^{3/4} = 2$, so raise each side to the 4/3 power to get $b = 2^{4/3}$. It rounds off to about 2.52.

11. (a) True.
    (b) False.
    (c) True. Be careful. The denominator is the log of the square root, not the square root of the log.
    (d) False. It is not true when a is negative. But change the right side to $2\log|a| - 2$ and it is true.
    (e) True.

12. (a) True.
    (b) True: it is $\log_2 5$, which equals (upon change to base 10) $\dfrac{\log 5}{\log 2}$.

13. (a) True.
    $\log_a uv = \log_a u + \log_a v$
    This is the first property of logarithms.
    (b) False.
    $(\log_a u) \div (\log_a v) = \log_a(u - v)$
    $2 = \log 100 \div \log 10 \neq \log(100 - 10) = \log 90 \approx 1.95$
    (c) True.
    $(\log_a b)(\log_b a) = (\log_a b)\left(\frac{\log_a a}{\log_a b}\right) = \log_a a = 1$. The key was changing the base of $\log_b a$ to base a.

14. (a) 2.838 (b) -0.8095 (c) 7.6485 (d) 0.87 (e) 4.7455 (f) -6.392 (g) 4.5495 (h) 0.69933

15. Only solutions where $x > 1$ will be valid.

    $\log\left(\frac{1}{x-1}\right) = 2$.
    $10^2 = \frac{x}{x-1}$.
    $100 = \frac{x}{x-1}$.
    $100x - 100 = x$, $99x = 100$, $x = 100/99$. That's OK; it is greater than 1.

16. (a) $x = 20$ (b) $x = 500/33$ or $x = 15\frac{5}{33}$ (c) $x = 25$

17. A valid solution must greater than 1/2.

    $\log_2\left(\frac{x}{2x-1}\right) = -3$.
    $2^{-3} = \frac{x}{2x-1}$.
    $\frac{1}{8} = \frac{x}{2x-1}$.
    $8x = 2x - 1$, and $x = -\frac{1}{6}$. But this proposed solution is not greater than 1/2.

    Conclusion: there is no solution to this equation.

18. Convert 80° to radians: $80° \times \frac{\pi \text{ radians}}{180°} = \frac{4\pi}{9}$.

    80 feet $= \frac{4\pi}{9} \times r$. Since $\frac{4\pi}{9}$ is a unitless ratio, do not include the word radians in this equation.

    $r = \frac{180}{\pi}$ feet, after solving.

19. (a) It is 2π radians per hour, as it goes around once each hour. That is one revolution per hour.
    For the degrees, 360 degrees per hour works out to 6 degrees per minute.
    (b) It is 2π radians per hour, thus $\pi/30$ radians per minute.

    4 feet is 48 inches.
    $s = r\theta = (48 \text{ inches}) \times \pi/30$ radians per minute, giving 1.6π inches per minute.

(c) It takes 12 hours for the hour hand to complete one revolution.
In one hour, the hour hand makes one-twelfth of a revolution. Its rotational velocity is therefore $\frac{2\pi}{12}$ radians per hour. That is 30 degrees per hour, which works out to 1/2 degree per minute.

(d) Subtract the angular velocities to get $5\frac{1}{2}$ degrees per minute.

20. $\cos 30° = \sqrt{3}/2 = 4/h$. So $h = 8/\sqrt{3} = \frac{8}{3}\sqrt{3}$ after rationalization.

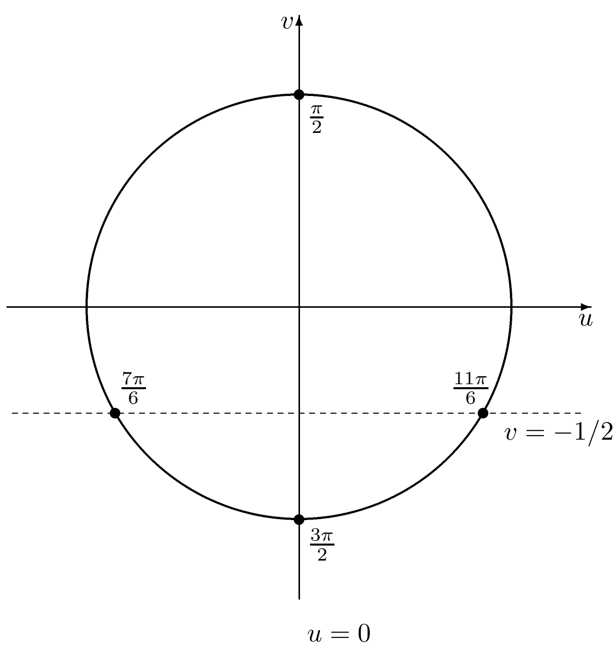
21. $\pi/6$ and $2\pi - \pi/6$. So $\pi/6$ and $11/\pi/6$. For the cosine, the primary solution is always the arccosine and the second one is either 2π minus the first solution or minus the first solution.

22. $41.99°$ and $180° - 41.99° = 138.01°$. For the sine, the first solution is obtained from the arcsine and a second solution is $180°$ minus the first.

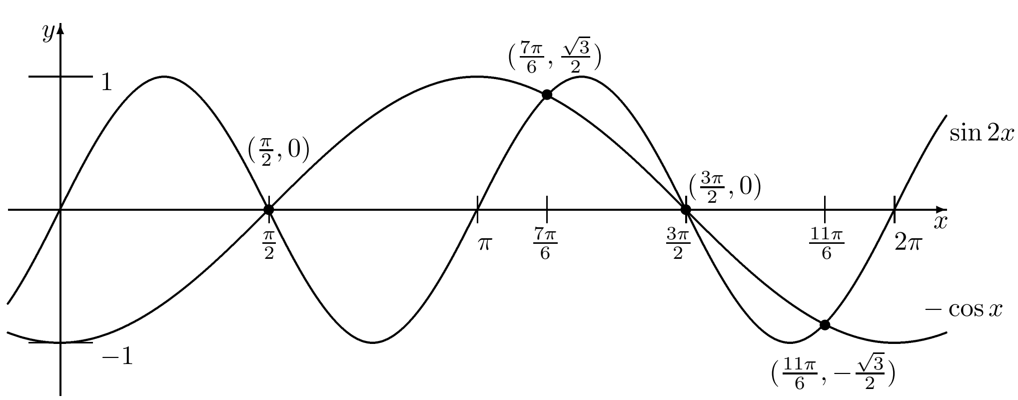
23. $105.07°$ and $360° - 105.07° = 254.93°$.

24. (a)
$$\begin{aligned} &2\sin x\cos x = -\cos x \\ &2\sin x\cos x + \cos x = 0 \\ &\cos x(2\sin x + 1) = 0 \end{aligned}$$

$$\begin{array}{lcl} \cos x = 0 & \text{or} & 2\sin x + 1 = 0 \\ x = \frac{\pi}{2} \text{ or } x = \frac{3\pi}{2} & & 2\sin x = -1 \\ & & \sin x = -1/2 \\ & & x = \frac{7\pi}{6} \text{ or } x = \frac{11\pi}{6} \end{array}$$



(b)



25. (a)

$$\begin{aligned}
&2\sin x\cos x = -\sin x\\
&2\sin x\cos x + \sin x = 0\\
&\sin x(2\cos x + 1) = 0
\end{aligned}$$

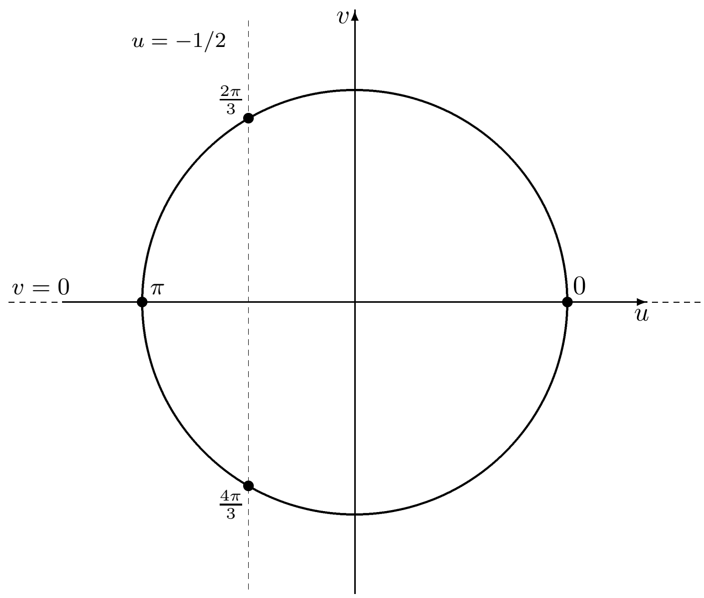
$\sin x = 0$ or $2\cos x + 1 = 0$

$x = 0$ or $x = \pi$ or $x = 2\pi$

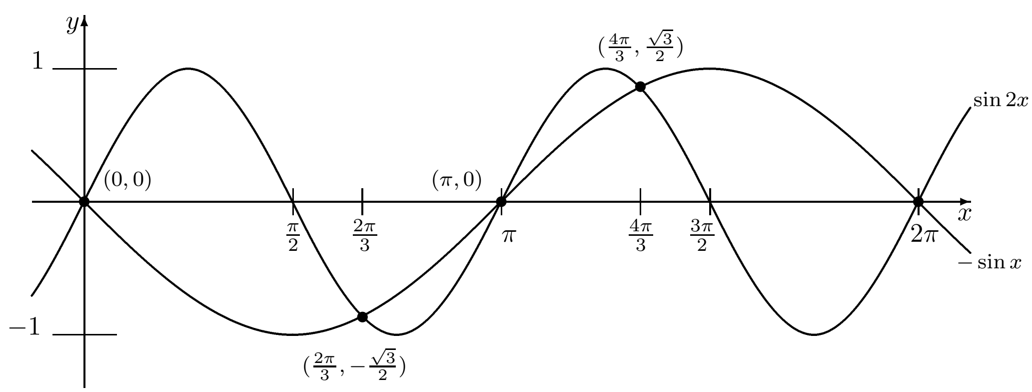
$2\cos x = -1$

$\cos x = -1/2$

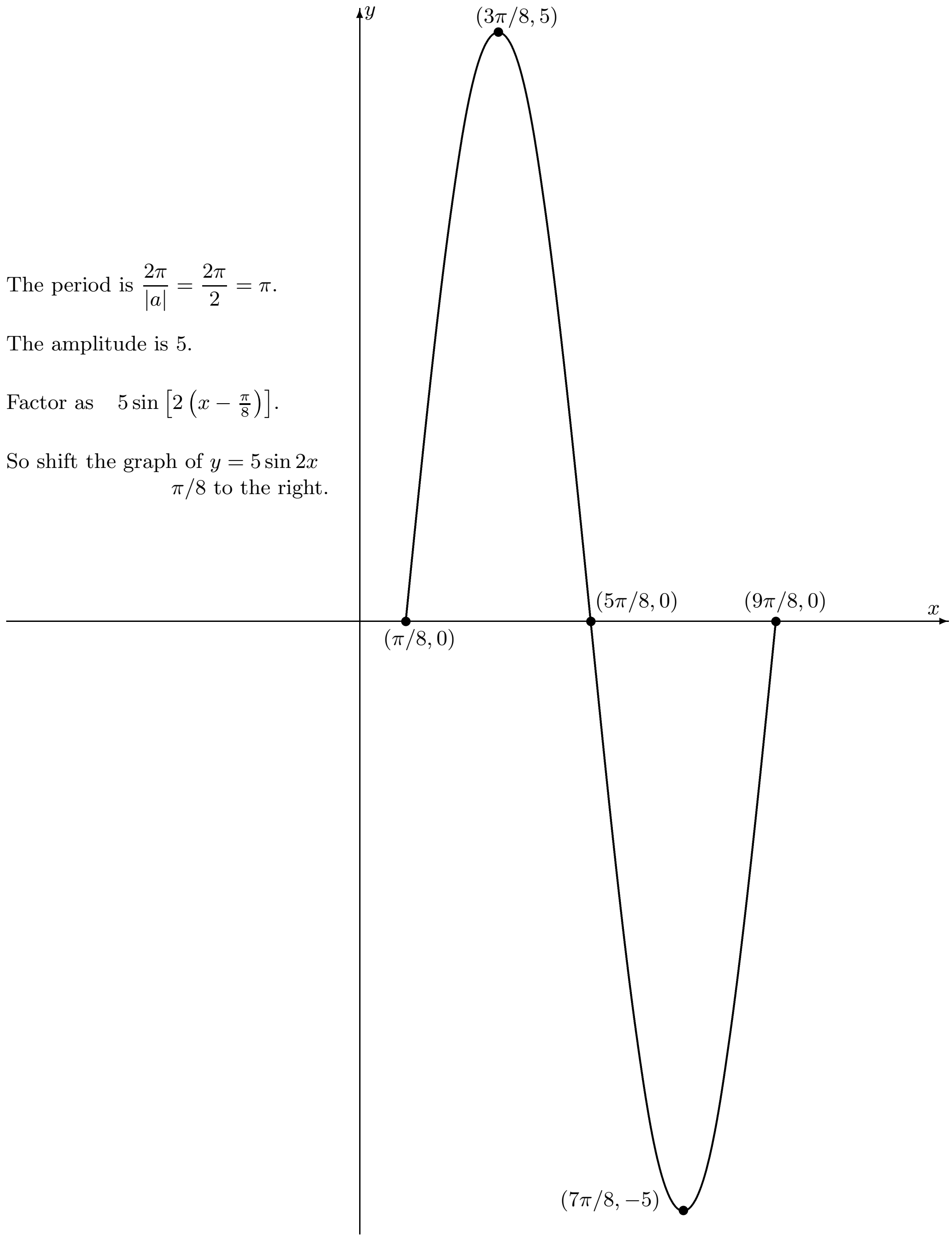
$x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$



(b)



26.



The period is $\dfrac{2\pi}{|a|} = \dfrac{2\pi}{2} = \pi$.

The amplitude is 5.

Factor as $\quad 5\sin\left[2\left(x - \frac{\pi}{8}\right)\right]$.

So shift the graph of $y = 5\sin 2x$ $\pi/8$ to the right.

27. The third side is 7 long, the angle opposite the side with length $5\sqrt{3}$ is $141.787°$, and the angle opposite the side with length 2 is $8.213°$. Note that these 3 angles add to $180.000°$.

The longest side is $5\sqrt{3}$, so the largest angle will be between the sides of lengths 2 and 7.

The shortest side is 2, so the smallest angle will between the sides of lengths 7 and $5\sqrt{3}$.

Work: $c^2 = 2^2 + (5\sqrt{3})^2 - 2(2)(5\sqrt{3})\cos 30° = 4 + 75 - 20\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 79 - 30 = 49$, so $c = 7$.

Largest angle: $\cos^2 = \frac{2^2+7^2-(5\sqrt{3})^2}{2\cdot 2\cdot 7} = \frac{53-75}{28} = \frac{-22}{28} = \frac{-11}{14}$.
Then the angle is the inverse cosine of $-11/14$, which comes out to about $141.787°$ on a calculator.

Best practice is to find the third angle by the law of cosines, then check that all three add up to $180°$. This will show if errors were made.

Smallest angle: $\cos^2 c = \frac{7^2+(5\sqrt{3})^2-2^2}{2(7)(5\sqrt{3})} = \frac{49+75-4}{70\sqrt{3}} = 120/(70\sqrt{3})$.

Then the angle is the inverse cosine of that expression, which comes out to about $8.213°$ on a calculator.

The three angles add up to exactly $180.000°$.