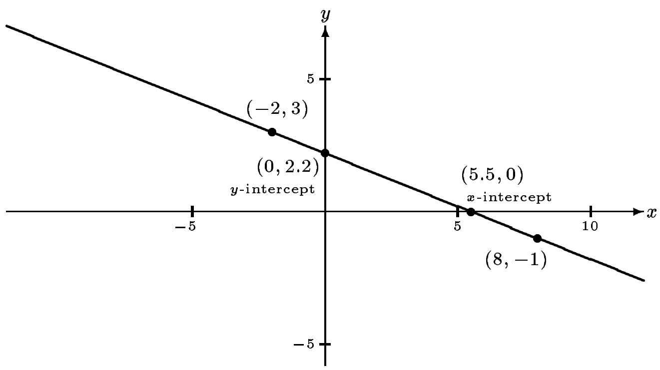
Answers to the Sample of Typical Final Examination Problems

Math 130 Precalculus for the May 19, 2017 Final Exam

1. $y = -\frac{2}{5}x + \frac{11}{5}$ or $y = -0.4x + 2.2$

Also $y - 3 = -\frac{2}{5}(x + 2)$ or $y + 1 = -\frac{2}{5}(x - 8)$

Also $2x + 5y = 11.$



Work: $m = \frac{-1-3}{8-(-2)} = \frac{-4}{10} = -2/5$ and $y - 3 = -\frac{2}{5}(x + 2).$

2. The equation is of the form $y = (-2/3)x + b$.

   Plug in the coordinates of the x-intercept $(6, 0)$ and solve.

   $0 = (-2/3)6 + b$, $(2/3)x = b$, and $b = 4$.

   So the equation is $y = (-2/3)x + 4$.

3. (a) $d = \sqrt{9^2 + 40^2} = \sqrt{1681} = 41.$

   The midpoint is at $\left(\frac{13.5+22.5}{2}, \frac{4.5+44.5}{2}\right) = (18, 24.5).$

   (b) The slope of the given line is $m = \frac{40}{9}$.

   The slope of any line perpendicular to it is $-\frac{9}{40}$.

   The perpendicular bisector goes through the midpoint. It is equation will be

$$y - 24.5 = \left(-\frac{9}{40}\right)(x - 18).$$

   In slope-intercept form, this equation is $y = \left(-\frac{9}{40}\right)x + 28.55.$

4. $(x - 11)^2 + (y - 10.5)^2 = 7.5$. The center is at $(11, 10.5)$ and the radius is 7.5.

   First find the midpoint of the diameter: $(\frac{5+17}{2}, \frac{6+15}{2})$. Ths will be the center.

   Then find the distance between the endpoints of the diameter:

   $\sqrt{(17-5)^2 + (15-9)^2} = \sqrt{144+81} = \sqrt{225} = 15.$

   The radius will be half of that, so 7.5 is the radius.

5. $\dfrac{f(2+h) - f(2)}{h} = \dfrac{(8 + 8h + 2h^2 + 6 + 3h - 1) - (2(2^2) + 3(2) - 1)}{h} = \dfrac{11h + 2h^2}{h} = 11 + 2h,\ h \neq 0.$

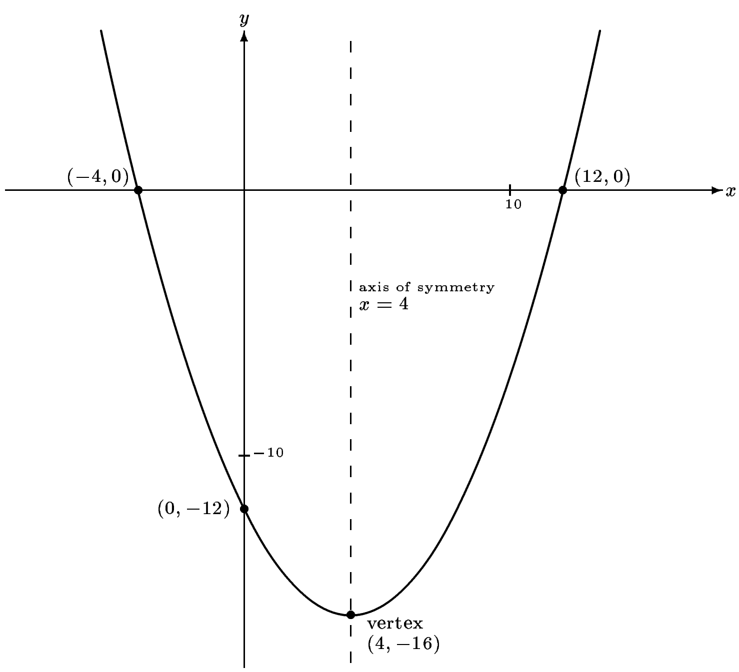
6. (a) f is odd because it simplifies to $\frac{3x^2-1}{x(x^2-1)}$, and when x has opposite sign, so does this fraction.
   (b) g is neither even nor odd because $g(3) = 4$ while $g(-3) = 2$.
   (c) h is odd because $h(a) = |a+3| - |a-3|$ and $h(-a) = |-a+3| - |-a-3| = |a-3| - |a+3| = -h(a)$. For support, show that $h(4) = 6$ and $h(-4) = -6$.
   (d) i is even because $i(x) = \sqrt{16 - x^2}$ showing that it is even by the rule of even powers and also that the domain is balanced, being $[-4, 4]$.
   (e) j is neither even nor odd because the domain, $x \geq 1$, is not balanced.

7. The graph is the well-known cubic.
   The new equation will be:
   $$y = (x+3)^3 - 3.$$

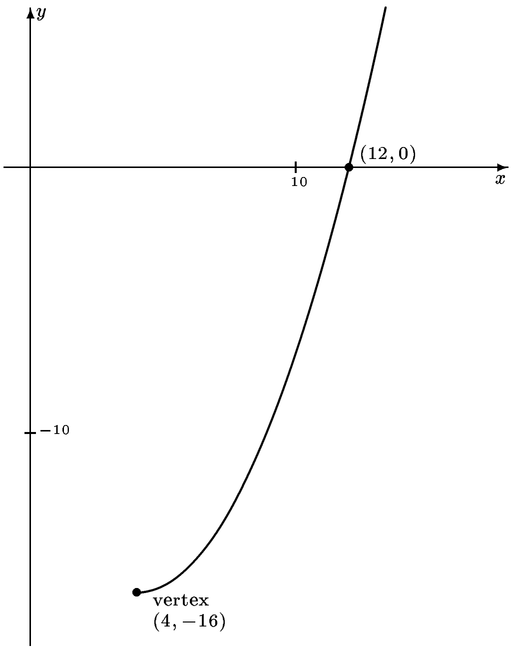
8. The graph is the well-known graph of the reciprocal of x.
   The new equation will be:
   $$y = \frac{1}{x+4} - 4.$$

9. (a) The range is $[0, \infty]$.
   The x-intercept is at $(-1, 0)$ and the y-intercept is at $(0, 1)$.
   (b) Yes, it is steadily increasing.
   To find the inverse: write $y = \sqrt{x+1}$, switch x and y to get: $x = \sqrt{y+1}$, then solve for y.
   $x^2 = y + 1$, so $y = x^2 - 1$. The formula for the inverse of f is: $f^{-1}(x) = x^2 - 1$.
   However, the domain of the inverse must be the same as the range of f. The inverse is $f^{-1}(x) = x^2 - 1$ for $x \geq 0$ only.
   The y-intercept is at $(0, -1)$ and the x-intercept is at $(1, 0)$. Just switch the x- and y-coordinates of the previous points, because the inverse is the reflection across the line $y = x$.

10. $y = \frac{1}{4}(x-4)^2 - 16.$
   The steps: $y = \frac{1}{4}(x^2 - 8x) - 12$, $y = \frac{1}{4}(x^2 - 8x + 16 - 16) - 12$, $y = \frac{1}{4}[(x-4)^2 - 16] - 12$.
   Be very careful to distribute to the dangling term, -16. The $\frac{1}{4}(-16)$ becomes -4.
   $$\text{So, the standard form is } y = \frac{1}{4}(x-4)^2 - 16.$$



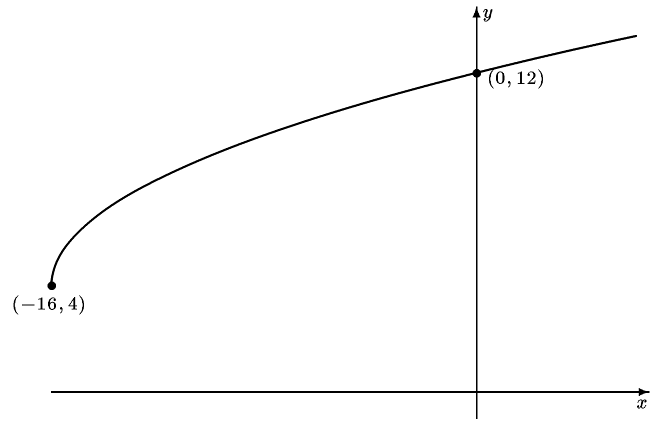
11. (a) $f(x) = \frac{1}{4}(x-4)^2 - 16$.



(b) It is one-to-one, so it has an inverse function.

The equation of the inverse function is $y = 2\sqrt{x+16} + 4$ for $x \geq -16$.

Switch variables and solve, using the standard form of the equation, from the previous problem.



This equation describes y as a function of x.

12. Let $x =$ the width and let $z =$ the length. The area $= xz$.
Since $x + z = 26$, we have $z = 26 - x$. Substituting $26 - x$ for z, we find that the area $= xz = x(26-x) = 26x - x^2 = -x^2 + 26x$. So for the area we have $f(x) = -x^2 + 26x$ with $a = -1 < 0$. That means that the parbola opens down and has a maximum value of $f(x)$.

$$k = c - \frac{b^2}{4a} = 0 - \frac{26^2}{4(-1)} = -\frac{576}{-4} = \frac{-576}{-4} = 169 \text{ square feet}$$

The maximum area is equal to the maximum value of $f(x)$, which equals the maximum value of y, which is called k. Remember that the quantity to be maximized is represented by y.
This problem could also be done by completing the square the long way.
We find that $f(x) = -x^2 + 26x = -(x^2 - 26x) = -(x^2 - 26x + 169 - 169) = -(x-13)^2 + 169$.
Thus $f(x) = -(x-13)^2 + 169$ and the vertex is $(13, 169) = (h, k)$.

13. Call the two numbers s and t.

$2s + 2t = 34$, so $2s = 34 - 2t$, and $s = 17 - t$.

The product is $p = st = (17-t)t = -t^2 + 17t$.

The product is a quadratic function of t. Graph it with t on the horizontal axis and p on the vertical axis.

The highest point is at the vertex. $k = c - \frac{b^2}{4a} = 0 - \frac{289}{-4} = \frac{289}{4} = 72\frac{1}{4}$.

The numbers are actually 8.5 and 8.5 and their product is 72.25.

14. Originally $y = \log_2 x$.

Now $y = \log_2(32x) = \log_2 32 + \log_2 x = 5 + \log_2 x$. The y ends up 5 larger, so the change in y is 5.

15. $\log_9(3\sqrt{3}) = \log_3(3\sqrt{3})/\log_3 9 = (\log_3 3 + \log_3 \sqrt{3})/2 = (1 + 1/2)(2) = (3/2)/2 = 3/4$.

16. $\log_b \frac{1}{2} = \log_b 1 - \log_b 2 = 0 - 3/4 = -3/4$.

    To evaluate $\log_2 b$, change the base to b and get $\log_2 b = \frac{\log_b b}{\log_b 2} = \frac{1}{3/4} = 4/3$.

    $b^{3/4} = 2$, so raise each side to the 4/3 power to get $b = 2^{4/3}$. It rounds off to about 2.52.

17. The points are (8, 3/2) and (32, 5/2).
    The slope will be $\frac{\Delta y}{\Delta x} = 1/24$.
    Plug in the point (8, 3/2) to get $y - 3/2 = (1/24)(x - 8)$.
    It simplifies to $y = (1/24)x + 3/2 - 1/3 = (1/24)x + 7/6$.

18. Only solutions where $x > 1$ will be valid.

    $\log\left(\frac{1}{x-1}\right) = 2$.
    $10^2 = \frac{x}{x-1}$.
    $100 = \frac{x}{x-1}$.
    $100x - 100 = x$, $99x = 100$, $x = 100/99$. That's OK; it is greater than 1.

19. (a) $x = 20$ (b) $x = 500/33$ or $x = 15\frac{5}{33}$ (c) $x = 25$

20. A valid solution must greater than 1/2.

    $\log_2\left(\frac{x}{2x-1}\right) = -3$.

    $2^{-3} = \frac{x}{2x-1}$.

    $\frac{1}{8} = \frac{x}{2x-1}$.

    $8x = 2x - 1$, and $x = -\frac{1}{6}$. But this proposed solution is not greater than 1/2.

    Conclusion: there is no solution to this equation.

21. (a) True.
    (b) False.
    (c) True.

22. (a) True.
    $\log_a uv = \log_a u + \log_a v$
    This is the first property of logarithms.
    (b) False.
    $(\log_a u) \div (\log_a v) = \log_a(u - v)$
    $2 = \log 100 \div \log 10 \neq \log(100 - 10) = \log 90 \approx 1.95$
    (c) True.
    $(\log_a b)(\log_b a) = (\log_a b)\left(\frac{\log_a a}{\log_a b}\right) = \log_a a = 1$. The key was changing the base of $\log_b a$ to base a.

23. (a) False. Try it with $a = 2$, $b = 4$, and $c = 8$.
    (b) True. Take $\log_b$ of each and show that the results are the same.

24. Just take the common log (base 10) of both expressions.
    The answers come out to $\log(a^{\log b}) = \log b \log a$ and $\log(b^{\log a}) = \log a \log b$.
    Because the log function is one-to-one, the conclusion is that the given expressions were equal.

25. About 566 bacteria. The one-hour growth factor is $\sqrt[4]{4} = \sqrt{2}$, and $400\sqrt{2} \approx 566$.

    By equations: $a^{\Delta t} = \frac{400}{100}$. But the change in time is 4 hours.

    So: $a^4 = 4$ and $a = 4^{1/4} = \sqrt{2}$.

    For the next value, 8 hours: $a^{\Delta t} = \frac{x}{400}$, and the change in time is 1 hour.

    So $x = 400 \times a^1 = 400\sqrt{2}$.

26. $a^1 = 320/200 = 1.6$. That means $a = 1.6$.

    Going from 2 hours to 3 hours, the change in time is again 1 hour, and a is known to be 1.6.

    $1.6^1 = \frac{x}{320}$, and $x = 320 \times 1.6 = 512$ bacteria.

    Of course 440 was incorrect; that would have been the case had the growth been linear.

27. 2.5π meters ≈ 7.85398 meters, because $150° = \frac{5\pi}{6}$ and $3\left(\frac{5\pi}{6}\right)$ meters $= \frac{5\pi}{2}$ meters.

28. 125 feet, because $150° = \frac{5\pi}{6}$ and $\frac{150}{\pi}\left(\frac{5\pi}{6}\right)$ feet $= 125$ feet after cancellation.

29. (a) It is 2π radians per hour, as it goes around once each hour. That is one revolution per hour. For the degrees, 360 degrees per hour works out to 6 degrees per minute.

    (b) It is 2π radians per hour, thus $\pi/30$ radians per minute.

    4 feet is 48 inches.
    $s = r\theta =$ (48 inches) $\times\, \pi/30$ radians per minute, giving 1.6π inches per minute.

    (c) It takes 12 hours for the hour hand to complete one revolution.
    In one hour, the hour hand makes one-twelfth of a revolution. Its rotational velocity is therefore $\frac{2\pi}{12}$ radians per hour. That is 30 degrees per hour, which works out to 1/2 degree per minute.

    (d) Subtract the angular velocities to get $5\frac{1}{2}$ degrees per minute.

30. $\sqrt{3}/2$; $\pi/6$ or $30°$; and $5\pi/6$ or $150°$. Then $1/2$; and $-\pi/3$ or $-60°$.

    (a) $\sin^2\theta + \cos^2\theta = 1$. Here $\sin^2 30° + \cos^2 30° = 1$, giving $(1/2)^2 + (\sqrt{3}/2)^2 = 1/4 + 3/4 = 1$. It holds.

    (b) $\arccos a + \arcsin a = \pi/2$. Here $\arccos\left(-\frac{\sqrt{3}}{2}\right) + \arcsin\left(-\frac{\sqrt{3}}{2}\right) = 150° + (-60°) = 90° = \pi/2$.

    (c) $\cos(\arccos a) = a$ for all a in $[-1, 1]$. Here (from above) $\arccos\left(\frac{-\sqrt{3}}{2}\right) = 150°$, so $\cos 150° = -\sqrt{3}/2$. That is true. The expressions are equal.

    (d) $\arccos(-a) = 180° - \arccos a$. Here $150° = 180° - 30°$. It holds.

31. The cosine is positive (having the same sign as the secant) and the sine is negative. That means the tangent will be negative.

    Use $1 + \tan^2 = \sec^2$, to find $1 + \tan^2 = 10$. So $\tan^2 = 9$, and $\tan = -3$, knowing that it is negative.

32. The sine is positive, because it has the same sign as the cosecant. So the cotangent is positive.

    $1 + \cot^2 = \csc^2$. That gives $1 + \cot^2 = 29/4$, and $\cot^2 = 25/4$. So $\cot\theta = 5/2$, it being the positive root.

33. (a) $\dfrac{1}{1 - 2r^2}$.

    (b) $\frac{\sqrt{2}}{2}(\sqrt{1 - r^2} + r)$.

    (c) $\pi - \arcsin r$.

34. (a) $\sqrt{2}/3$ (b) $-\sqrt{7}/3$ (c) $\sqrt{7}/3$ (d) $-\sqrt{7}/3$ (e) $\sqrt{2}/3$

35. $\cos 30° = \sqrt{3}/2 = 4/h$. So $h = 8/\sqrt{3} = \frac{8}{3}\sqrt{3}$ after rationalization.

36. $\pi/6$ and $2\pi - \pi/6$. So $\pi/6$ and $11/\pi/6$. For the cosine, the primary solution is always the arccosine and the second one is either 2π minus the first solution or minus the first solution.

37. $41.99°$ and $180° - 41.99° = 138.01°$. For the sine, the first solution is obtained from the arcsine and a second solution is $180°$ minus the first.

38. $105.07°$ and $360° - 105.07° = 254.93°$.

39. (a) $\cos x(1 + \tan x)(1 - \tan x) = \cos x(1 - \tan^2 x) = \cos x(1 - (\sec^2 x - 1)) = \cos x(2 - \sec^2 x) =$

$$2\cos x - \frac{1}{\cos x} = \frac{2\cos^2 x - 1}{\cos x}.$$ This answer could also be given as $\frac{\cos 2x}{\cos x}$.

(b) $\tan x \cos^2 x = (\sin x / \cos x)\cos^2 x = \sin x \cos x = \frac{1}{2}\sin 2x$.

(c) $\cos^4 x - \sin^4 x = (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) = (1)(\cos 2x) = \cos 2x$

(d) $\frac{1+\cot^2 x}{\sin x} = \frac{\csc^2 x}{\sin x} = \csc^2 x\left(\frac{1}{\sin x}\right) = \csc^2 x(\csc x) = \csc^3 x$.

(e) $\frac{\sec x}{\csc x} = \left(\frac{1}{\cos x}\right)\left(\frac{1}{\csc x}\right) = \left(\frac{1}{\cos x}\right)\sin x = \frac{\sin x}{\cos x} = \tan x$.

(f) $\frac{\sec x}{\sin x} = \sec x\left(\frac{1}{\sin x}\right) = \left(\frac{1}{\cos x}\right)\left(\frac{1}{\sin x}\right) = \frac{1}{\sin x \cos x} = \frac{1}{(1/2)\sin 2x} = 2\csc 2x$.

40. $(2\cos^2\theta - 1) - 2\cos^2\theta = -1$.

or

$(1 - 2\sin^2\theta) - 2(1 - \sin^2\theta) = 1 - 2\sin^2\theta - 2 + 2\sin^2\theta = -1$.

Check it by assigning a value to θ, say $\pi/3$, and evaluating.

$\cos 2\pi/3 = -1/2$ and $\cos\pi/3 = 1/2$, so the expression ends up as

$-1/2 - 2(1/2)^2 = -1/2 - 2/4 = -1$, and it checks.

41. An equivalent equation in terms of $\cos x$ is obtained by substituting $2\cos^2 x - 1$ for $\cos 2x$.

$3(2\cos^2 x - 1) = 2\cos^2 x$, so $4\cos^2 = 3$, $\cos^2 x = 3/4$, and $\cos x = \pm\sqrt{3}/2$.

For $+\sqrt{3}/2$ the answers are $\pi/6$ and $-\pi/6$, which becomes $11\pi/6$ in the interval specified.

For $-\sqrt{3}/2$ the primary solution is $5\pi/6$, the supplement of $\pi/6$. The other solution is $-5\pi/6$, which becomes $7\pi/6$ in the interval specified.

Answers: $\pi/6$, $5\pi/6$, $7\pi/6$, and $11\pi/6$.

42. (a)

$$2\sin x\cos x = -\cos x$$
$$2\sin x\cos x + \cos x = 0$$
$$\cos x(2\sin x + 1) = 0$$

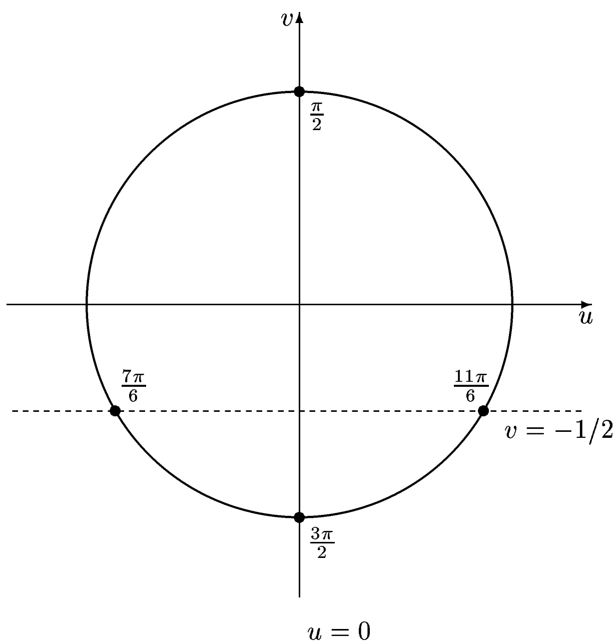
$\cos x = 0$ or $2\sin x + 1 = 0$

$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$

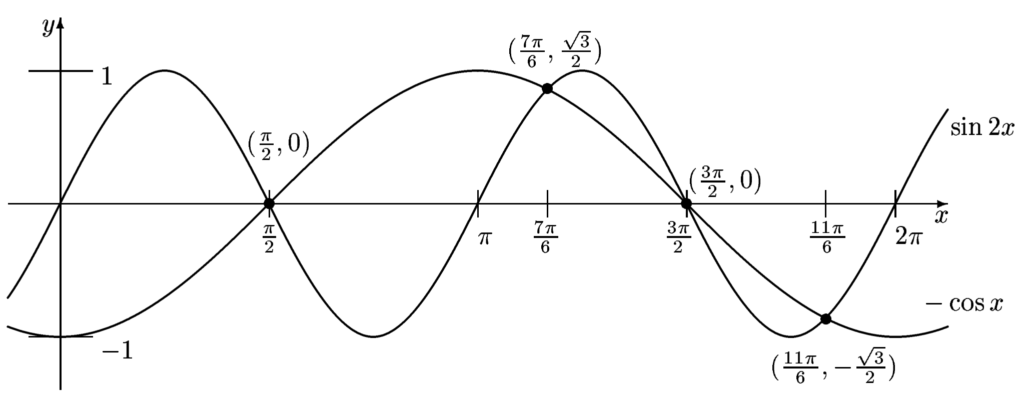
$2\sin x = -1$

$\sin x = -1/2$

$x = \frac{7\pi}{6}$ or $x = \frac{11\pi}{6}$



(b)



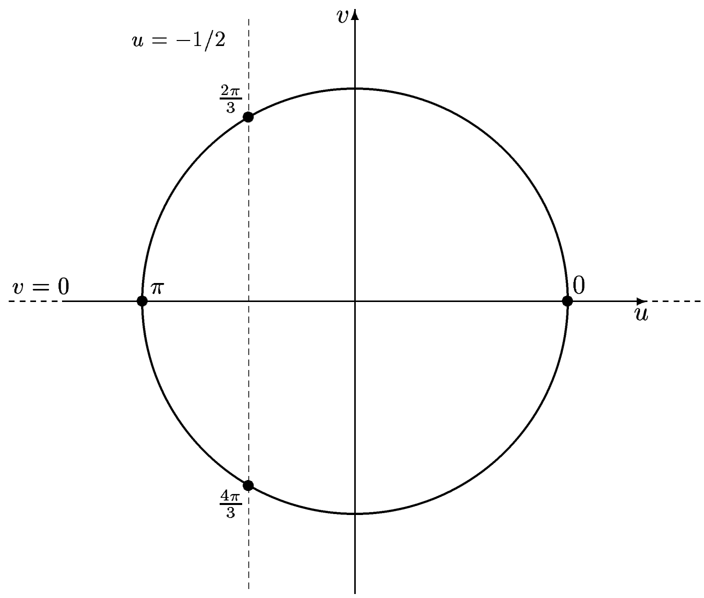
43. (a)

$$2\sin x\cos x = -\sin x$$
$$2\sin x\cos x + \sin x = 0$$
$$\sin x(2\cos x + 1) = 0$$

$\sin x = 0$
$x = 0$ or $x = \pi$ or $x = 2\pi$

or

$2\cos x + 1 = 0$
$2\cos x = -1$
$\cos x = -1/2$
$x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$



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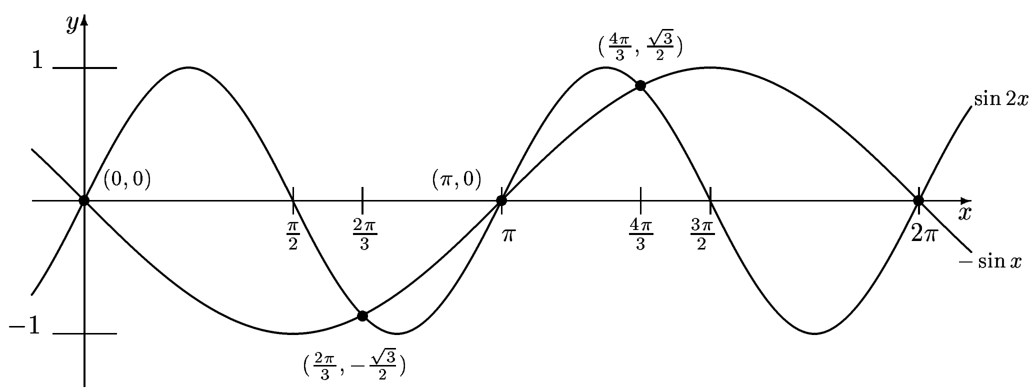


44. (a) The formula is $\sin \frac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}}$.

    Here it comes out as $+\sqrt{\frac{1-\sqrt{3}/2}{2}} = \sqrt{\frac{2-\sqrt{3}/2}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}$.

    (b) It will be $\sin 45° \cos 30° - \cos 45° \sin 30° = (\sqrt{2}/2)(\sqrt{3}/2) - (\sqrt{2}/2)(1/2) = 1/4(\sqrt{6} - \sqrt{2})$.

    (c) They are different but equivalent.
    One way to show this is to find approximate decimals for each of them.
    A more formal method is to square them both to get: $\frac{2-\sqrt{3}}{4}$ and $\frac{1}{16}(8-2\sqrt{12})$. But, since $\sqrt{12} = 2\sqrt{3}$, the exact expressions are proved to be equal. (Of course both expressions were positive.)

45. First find the cosine of 135°, which is $-\sqrt{2}/2$.

    By the half-angle formula for the cosine (this angle is one half of 135°), it is $+\sqrt{\frac{1+(-\sqrt{2}/2}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$.

    Use a calculator to evaluate the above expression and see that it is indeed the cosine of $67\frac{1}{2}^{\circ}$. These are only approximate values, about 0.3826834, so they do not answer the question as given, but they certainly support the result.

46. By the difference formula for cosine, $\cos 15° = \cos 45° \cos 30° + \sin 45° \sin 30° = (\sqrt{2}/2)(\sqrt{3}/2) + (\sqrt{2}/2)(1/2) = 1/4(\sqrt{6} + \sqrt{2})$.

    Verification on a calculator: the value of the above radical expression is approximately 0.9659, as is the cosine of 15 degrees. This supports the result. But the question itself required the exact value.

47. The double angle for the cosine is $\cos 2\theta = 2\cos^2\theta - 1$.

    So $\cos 2u = 2(-2/5)^2 - 1 = 8/25 - 1 = -17/25$.

    And, since $4u$ is double $2u$, $\cos 4u = 2(-17/25)^2 - 1 = 2(289/625) - 1 = 578/625 - 1 = -47/625$.

48. The double angle for the cosine is $\cos 2\theta = 2\cos^2\theta - 1$.

    So $\cos 2u = 2(-2/3)^2 - 1 = 8/9 - 1 = -1/9$.

    And, since $4u$ is double $2u$, $\cos 4u = 2(-1/9)^2 - 1 = 2(1/81) - 1 = 2/81 - 1 = -79/81$.

49. The double angle for the cosine is $\cos 2\theta = 2\cos^2\theta - 1$.

    So $\cos 2u = 2(\sqrt{6}/6)^2 - 1 = 2(6/36) - 1 = 2(1/6) - 1 = 1/3 - 1 = -2/3$.

    And, since $4u$ is double $2u$, $\cos 4u = 2(-2/3)^2 - 1 = 2(4/9) - 1 = 8/9 - 1 = -1/9$.

50. The third side is 1 long, the angle opposite the side with length $\sqrt{3}$ is 120°, and the third angle must be 30° because the triangle is isosceles. Note that these 3 angles add to 180°.

    Work: $c^2 = 1^2 + (\sqrt{3})^2 - 2(1)(\sqrt{3})\cos 30° = 1 + 3 - 2\sqrt{3}(\frac{\sqrt{3}}{2}) = 1 + 3 - 3 = 1$, so $c = 1$.

    Then use the fact that this triangle is isosceles to get the other two angles immediately.

51. The third side is 7 long, the angle opposite the side with length $5\sqrt{3}$ is 141.787°, and the angle opposite the side with length 2 is 8.213°. Note that these 3 angles add to 180.000°.

    The longest side is $5\sqrt{3}$, so the largest angle will be between the sides of lengths 2 and 7.
    The shortest side is 2, so the smallest angle will between the sides of lengths 7 and $5\sqrt{3}$.

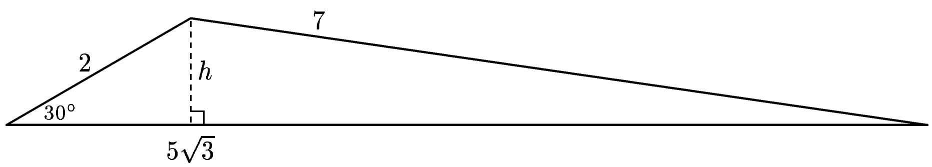
    Work: $c^2 = 2^2 + (5\sqrt{3})^2 - 2(2)(5\sqrt{3})\cos 30° = 4 + 75 - 20\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 79 - 30 = 49$, so $c = 7$.

    Largest angle: $\cos^2 = \frac{2^2+7^2-(5\sqrt{3})^2}{2\cdot 2\cdot 7} = \frac{53-75}{28} = \frac{-22}{28} = \frac{-11}{14}$.
    Then the angle is the inverse cosine of $-11/14$, which comes out to about 141.787° on a calculator.

    Best practice is to find the third angle by the law of cosines, then check that all three add up to 180°. This will show if errors were made.

    Smallest angle: $\cos^2 c = \frac{7^2+(5\sqrt{3})^2-2^2}{2(7)(5\sqrt{3})} = \frac{49+75-4}{70\sqrt{3}} = 120/(70\sqrt{3})$.

    Then the angle is the inverse cosine of that expression, which comes out to about 8.213° on a calculator.
    The three angles add up to exactly 180.000°.



52. The cosine and the cotangent are both negative. Since the cotangent is cosine divided by the sine, the sine must be positive. The terminal point for θ must lie in the second quadrant and the sine will be positive.
    $1 + \cot^2 = \csc^2$, so $\csc^2\theta = 1 + (-24/7)^2 = 1 + 576/49 = 625/49$. So the csc of θ is $25/7$, positive along with the sine.
    The sine is the reciprocal of the cosecant. The sine of θ is $7/25$.

53. The tangent is positive and the sine is negative. That puts the terminal point in the third quadrant, and the cosine will be negative.
    $1 + \tan^2\theta = \sec^2\theta$. Here $\sec^2\theta = 1 + 13/36 = 49/36$.
    The secant has the same sign as the cosine. Taking the square root of both sides gives $\sec\theta = -7/6$.
    The cosine is the reciprocal of the secant. Answer: $\cos\theta = -6/7$.

54. (a) $-11/16$
    (b) $\sqrt{2}/8 - \sqrt{30}/8$
    (c) $\frac{1}{4}\sqrt{6}$ and $\frac{1}{4}\sqrt{10}$.