Sample of Typical Final Examination Problems

Math 130 Precalculus for the December 16, 2016 Final Exam

No books, notes, or graphing calculators; scientific calculators permitted.
Show all work with at least four-decimal-place accuracy.

1. Find an equation of the line passing through the points $(8, -1)$ and $(-2, 3)$. Sketch the line and label both intercepts with their coordinates.

2. Graph the equation $x = \sqrt{1 - y^2}$. Label all intercepts with their coordinates.
   - Does this equation describe y as a function of x?
   - In which quadrants does this graph lie?
   - Describe in words the shape of this graph.
   - Is this graph symmetric across the x-axis, the y-axis, or the line $y = x$? Is it symmetric through the origin?
   - Is the inverse of the relation described by this equation a function?
   - Explain how this graph could be used as a visual representation of the arcsine on the unit circle.

3. When $f(x) = 2x^2 + 3x - 1$ and $h \neq 0$, find $\dfrac{f(x+h) - f(x)}{h}$ and simplify the result.

4. In each case decide whether the function with the given rule is even, odd, or neither. Explain your reasoning or support your answer.

   (a) $f(x) = \dfrac{1}{x-1} + \dfrac{1}{x} + \dfrac{1}{x+1}$

   (b) $g(x) = |x + 1|$

   (c) $h(x) = |x + 3| - |x - 3|$

   (d) $i(x) = \sqrt{(4 - x)(4 + x)}$

5. For $f(x) = \sqrt{x}$ and $g(x) = x^4 - 6x^2 + 9$, find and simplify $f \circ g$, $g \circ f$, and f^{-1}. The simplified results will have no square roots.

   Find the domain of each function, the domain of each composite function, and the domain of f^{-1}. Only consider them as functions over the real numbers with real-valued outputs.

6. Let $f(x) = \dfrac{x}{x+1}$, and let $g(x) = \dfrac{1}{x} - 1$.

   Find and simplify: $(f \circ g)(x)$, $(g \circ f)(x)$, $(f \circ f)(x)$, $(g \circ g)(x)$, and $f^{-1}(x)$; and state the domain for each of the seven cases (f itself, g itself, the four compositions, and the inverse).

7. Determine whether $f(x) = \dfrac{4}{-5x + 3}$ has an inverse function. If it does, find the inverse function.

8. True or false: the equations $y = 1 - x$ and $x = 1 - y$ will have the same graph.

9. Write the quadratic function $f(x) = \frac{1}{4}x^2 - 2x - 12$ in standard form and sketch its graph.

   Plot the vertex, axis of symmetry, and all intercepts, labelling with both coordinates or the equation.

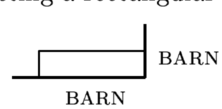
10. (a) Graph the equation $y = \frac{1}{4}x^2 - 2x - 12$ when $x \geq 4$.

      i. Does this equation describe y as a function of x?

    (b) Is the inverse of the relation described in part (a) a function?

      i. If so, find a closed-form formula for the inverse function and graph it.

11. Find the largest area that a farmer can enclose by constructing a rectangular pen from 26 feet of fencing, if he uses a corner of his barn for two walls of the pen.



12. Consider the result of taking three times the square of one real number and then adding twice a second real number, when these two real numbers add to 20.

Find two such numbers for which the result is as small as possible.

Hints:

1) Avoid using the variable y for either of the two numbers.

2) Find a formula of the result as a function of the first number.

3) Do not assume that the numbers must be integers.

13. (a) Consider the functions f: defined by the equation $y = 4^x$; and
g: defined by the equation $y = \log_4 x$.

For the function f: list the ordered pairs in the solution set for the x values:
$-2, -1, -1/2, 0, 1/2, 1, 1\frac{1}{2}$, and 2.

For the function g: list the ordered pairs in the solution set for the x values:
$\frac{1}{16}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8$, and 16.

What is special about the eight ordered pairs in each list when proper comparison is made with the other list? What does this tell you about these ordered pairs, when they are paired up in an obvious way.

Next plot the ordered pairs on the x,y-plane. Then decide how the 8 points on each graph are related to the 8 points on the other graph. Explain using the language of reflection.

(b) Graph $y = 4^x$ for $-2 \leq x \leq 2$.
Then graph on the same coordinate system $y = \log_4 x$ for $\frac{1}{16} \leq x \leq 16$.
For each graph, plot with exact coordinates all intercepts and the points for the x values listed in part (a).
Do these graphs have the same shape?
If they do have the same shape, describe using—the language of reflection and/or symmetry—how one is related to the other.
Also describe the asymptote for each graph.

14. Let $w = \log_2 a$.

Find an expression in terms of w for:

(a) $\log_2 a^4$
(b) $\log_2(4a^2)$
(c) $\log_2(4a)^2$
(d) $\log_2(\sqrt[4]{a})$
(e) $[\log_2 4a]^2$
(f) $\sqrt{\log_2 a^2}$, assuming that w is not negative.

15. In each of parts (a) through (h), approximate the logarithm, using the properties of logarithms, given $\log_b 2 \approx 1.098$, $\log_b 3 \approx 1.740$, and $\log_b 5 \approx 2.5495$.

(a) $\log_b 6$
(b) $\log_b \frac{3}{5}$
(c) $\log_b 125$
(d) $\log_b \sqrt{3}$
(e) $\log_b 20$
(f) $\log_b(4b)^{-2}$
(g) $\log_b(5b^2)$
(h) $\log_b \sqrt[3]{2b}$

16. Solve algebraically:

(a) $\log x + \log(x - 15) = 2$.
(b) $\log x - \log(x - 15) = 2$.
(c) $\log_8(x + 1) - \log_8(x - 3) = \frac{1}{3}$.
(d) $\log 24x - \log(1 + \sqrt{x}) = 2$.

17. Solve algebraically: $\log_2 x + \log_2(1-3x) = -4$.

Hint: to solve the equation found after applying some log rules, just use the quadratic formula.

18. First decide in which intervals all valid solutions must lie.

Then solve for x.

$$\log_2 x + \log_2(1-2x) = -3.$$

Check your solutions in the original equation.

19. First decide in which intervals all valid solutions must lie.

Then solve for x.

$$\log_2(-x) - \log_2(1-2x) = -3.$$

Check your solutions in the original equation.

20. True or false:

(a) $\log(3.4 \times 13.4) = \log 3.4 + \log 13.4$.

(b) $\dfrac{\log \frac{1}{2}a - \log a}{\log\left(\frac{1}{2}\right)^{1/2}} = 2$ for all a such that $a > 0$.

(c) $\log\left(\frac{1}{100}a^2\right) = 2\log a - 2$ for a any non-zero real number.

(d) The log of the quotient equals the difference of the logs.

21. True or false: since $(-4)^1 = (-4)$, $\log_{-4}(-4) = 1$.

22. Decide if each statement is true or false. Then justify your answer by writing an equation.

(a) Multiplying two numbers then taking the log gives the same result as taking each log and then adding them.

(b) Taking the logs of two numbers then dividing those two logs gives the same result as subtracting the two numbers then taking the log of that difference.

(c) The product of $\log_a b$ and $\log_b a$ is always equal to 1.

23. True or false. (In each part assume that all three numbers are positive.)

(a) $a^{\log_b c}$ and $b^{\log_a c}$ are equal.

(b) $a^{\log_b c}$ and $c^{\log_b a}$ are equal.

24. The number of bacteria present in a culture is increasing according to the law of exponential growth. After 3 hours there are 100 bacteria and after 7 hours there are 400 bacteria. How many bacteria will there be after 8 hours?

25. A population of fruit flies is increasing according to the law of exponential growth. At time 2 hours there are 2 pounds of flies and at time 32 hours there are 32 pounds of flies.

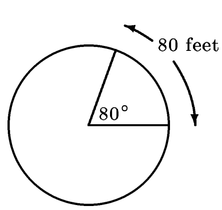
(a) Find the exact value of the doubling time. (No calculator is necessary.)

(b) True or false: at time 8 hours there were exactly 8 pounds of fruit flies.

(c) If false, about how many pounds of fruit flies were there at time 8 hours (to the nearest three-decimal accuracy or as an exact radical expression).

(d) At exactly what time will there be 64 pounds of fruit flies?

26. A group has a banner that is 80 feet long. They wish to display it in the form of an arc of a circle that has angular measure of 80°. What is the radius of the circle needed for this layout? The answer may be left as $\frac{N}{\pi}$ feet.



27. A right triangle has an acute angle θ with $\sec\theta = \frac{8}{7}$. Find the exact values of the other five trigonometric functions of θ, in fractional form. Some of the expressions will involve square roots; do not converst the square roots to decimals.

    Then find the exact values of $\sec(90° - \theta)$ and of $\csc^2\theta - 1$, also in fractional form.

    Hint. First sketch a right triangle corresponding to that secant. Next use the Pythagorean Theorem to determine the third side. Then find the other five trigonometric functions of θ.

    For the other two values, use the appropriate trigonometric identities.

28. Let $\sin\theta = r$, where $-1 \le r \le 1$.

    Find, in terms of r:

    (a) $\sec 2\theta$. Assume that $r \ne \pm\sqrt{2}/2$.
    (b) $\sin(\theta + \frac{\pi}{4})$, assuming that θ is between $-\pi/2$ and $\pi/2$.
    (c) An expression that gives a value for θ in the interval $\pi/2 \le \theta < 3\pi/2$.
    Hint: This will involve an inverse trig function of r.

29. Let $\theta = \arcsin r$.

    Find, in terms of r:

    (a) $\tan\theta$.
    (b) $\cos^2(\pi - \theta)$.

30. Suppose that θ is between 0 and π and that $\cos\theta = \frac{7}{25}$.

    Find the exact value of $\cos\left(\theta + \frac{\pi}{4}\right)$, in the form of a decimal or rational multiple of $\sqrt{2}$.

31. Suppose that θ is between 0 and π and that $\cos\theta = \frac{3}{5}$.

    Find an exact expression for $\sin\left(\theta + \frac{\pi}{3}\right)$, leaving $\sqrt{3}$ in radical form.

32. Given that $\cos u = -4/5$ with $\pi/2 < u < \pi$, find the exact values of $\sin 3u$ and $\cos 3u$ using the double-angle formulas and the sum formulas, as needed. Don't use a calculator. *Hint:* use fractions, not decimals.

33. An angle θ is between $\frac{3\pi}{2}$ and 2π, and $\cos\theta = 4/5$.

    (a) Find $\cos\frac{1}{2}\theta$, not as a decimal, but in radical form with a rationalized denominator.
    (b) Find $\cos(\theta + \frac{\pi}{4})$. Leave the answer as a rational mutiple of $\sqrt{2}$.

34. Let $\theta = \arcsin\left(\frac{24}{25}\right)$.

    Find $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$, each as an exact rational or decimal.

    Suggestion: first find $\cos\theta$.

35. Let $\theta = \arccos\left(\frac{1}{8}\right)$.

    Find $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$, both as exact rational or decimal numbers.

    Do not use a calculator.

36. An angle θ is between 0 and $\pi/2$, and $\cos\theta = 7/9$.

    Find the exact values of $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$. Radical form is OK; do not use a calculator.

37. An angle θ is between π and $3\pi/2$, and $\cos\theta = -7/9$.

    (a) Find $\cos\frac{1}{2}\theta$. Find the exact fraction; do not use a calculator.
    (b) Is $\sin\frac{1}{2}\theta$ positive or negative?

38. An angle θ is between π and $\frac{3\pi}{2}$, and $\cos\theta = -0.82$.

    Find the exact value of $\cos\frac{1}{2}\theta$, without using a calculator.

39. An angle θ is between 0 and 2π, and $\cos\theta = -0.62$.

    (a) Find the exact value of $\sin\frac{1}{2}\theta$, without using a calculator.
    (b) With the use of a calculator, find all possible values of θ in degrees. Approximations are fine.

40. Each of angles α and β are between 0 and $\pi/2$, with $\cos\alpha = 1/4$ and $\cos\beta = 7/8$.
Let $\theta = \alpha + \beta$.

(a) Which is greater: $\sin\alpha$ or $\sin\beta$? By what factor? Which sine was expected to be greater?
(b) Find the sine and cosine of θ.
(c) Find the sine of $\frac{1}{2}\theta$.
(d) Find $\cos 4\theta$.
(e) Comparing the sine and cosine of α to the sine and cosine of θ, what can be concluded about the relationship of α and θ?

41. Evaluate the expression

$$\frac{\sin s}{s},$$

for any very small real number of your choosing.
State your answer to the nearest 6 decimal places.

42. Simplify and reduce to an expression that contains at most one trig function.

(a) $\cos x(1 + \tan x)(1 - \tan x)$
(b) $\tan x \cos^2 x$
(c) $\cos^4 x - \sin^4 x$
(d) $\dfrac{1 + \cot^2 x}{\sin x}$
(e) $\dfrac{\sec x}{\csc x}$
(f) $\dfrac{\sec x}{\sin x}$

43. Find the numerical value of $\cos 2\theta - 2\cos^2\theta$.

44. Simplify $\cos\frac{1}{2}\theta \cdot \sin\frac{1}{2}\theta$ to a an expression with a single trig function.

45. Prove that $\dfrac{\sin\frac{1}{2}\theta}{\frac{1}{2}\sin\theta}$ is equivalent to $\sec\frac{1}{2}\theta$ as long as θ is not an integer multiple of π.

46. Find all solutions of $3\cos 2x = 2\cos^2 x$ in the interval $[0, 2\pi)$. (Get the algebraically-assisted, exact result.)

47. (a) Find all solutions with $0 \le x \le 2\pi$ for $\sin 2x = -\cos x$.
(b) Graph $\sin 2x$ and $-\cos x$ on the same axes and indicate on your sketch the points corresponding to the solutions in part (a).

48. (a) Find all x between 0 and 2π for which $\sin 2x = -\sin x$.
(b) Sketch the graphs of $\sin 2x$ and $-\sin x$ on the same axes, indicating on your sketch the points corresponding to the solutions in part (a).

49. Use the Law of Cosines to solve the triangle with sides of lengths 1 and $\sqrt{3}$, and an included angle of 30° between them.

50. Use the Law of Cosines to solve the triangle with sides of lengths 2 and $5\sqrt{3}$, and an included angle of 30° between them. (Round angles to three decimal places.)

Hint. After finding the third side ask yourself: Which is the longest side? So the largest angle will be between which two sides? Which is the shortest side? So the smallest angle will be between which two sides?

Find the exact area of this triangle. Leave it as multiple of $\sqrt{3}$.

Hint. Drop a perpendicular from the opposite vertex to the side of length $5\sqrt{3}$.

51. Find the period and the amplitude of

$$y = 5\sin\left(2x - \frac{\pi}{4}\right).$$

Graph one period. Label with coordinates the endpoints of that period, the highest and lowest points, and all intercepts in that period.

State the phase fraction: the portion of a period that the graph was translated right (+) or left (−).

It might be less confusing with the 2 factored out of the expression in the parentheses.

52. Assume that the cosine of $75.5225°$ is $1/4$. (This is actually an approximation.)

Find without a calculator, using the sum, double-angle, and half-angle formulas:

(a) the cosine of $226.5675°$ as a rational number.
Suggestion: first find the sine of $75.5225°$, and then find sine and cosine of $151.0450°$.

(b) the cosine of $120.5225°$, leaving it in radical form.

(c) the sine and the cosine of $\frac{1}{2}(75.5225°)$, simplified to rational multiples of $\sqrt{6}$ and $\sqrt{10}$.