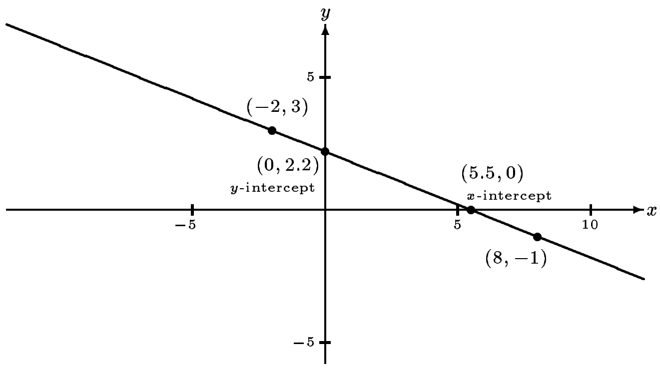
# Answers to the Sample of Typical Final Examination Problems

Math 130 Precalculus for the December 16, 2016 Final Exam

1. $$y = -\frac{2}{5}x + \frac{11}{5} \quad \text{or} \quad y = -0.4x + 2.2$$

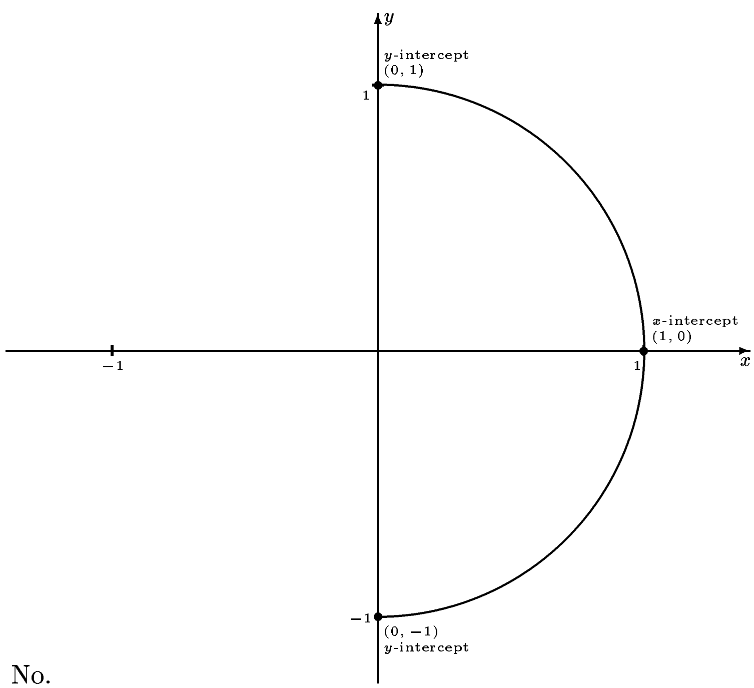
Also $$y - 3 = -\frac{2}{5}(x+2) \quad \text{or} \quad y + 1 = -\frac{2}{5}(x-8)$$

Also $$2x + 5y = 11.$$




*Work:* $m = \frac{-1-3}{8-(-2)} = \frac{-4}{10} = -2/5$ and $y - 3 = -\frac{2}{5}(x+2)$.

2.




- No.
- In the first and fourth quadrants only, as $x$ is the positive square root and $y$ could be positive or negative.
- It is the right unit semicircle.
- The graph has symmetry across the $x$-axis, but none of the others.
- Yes. It is one-to-one. The graph clearly passes the horizontal line test.
- This graph is the defining picture. The $y$-coordinate on this semicircle represents the input to the arcsine function, and the length of the directed arc from the point $(1, 0)$ to the point $(x, y)$ represents the output of the arcsine function.

3. $\frac{f(x+h) - f(x)}{h} = \frac{(2x^2 + 4xh + 2h^2 + 3x + 3h - 1) - (2x^2 + 3x - 1)}{h} = \frac{4xh + 2h^2 + 3h}{h} = 4x + 2h + 3, \ h \neq 0.$

4. (a) $f(x)$ is odd because it simplifies to $\frac{3x^2-1}{x(x^2-1)}$, and when $x$ has opposite sign, so does this fraction.
   (b) $g(x)$ is neither even nor odd because $g(3) = 4$ while $g(-3) = 2$.
   (c) $h(x)$ is odd because $h(a) = |a+3|-|a-3|$ and $h(-a) = |-a+3|-|-a-3| = |a-3|-|a+3| = -h(a)$. For support, show that $h(4) = 6$ and $h(-4) = -6$.
   (d) $i(x)$ is even because $i(x) = \sqrt{16-x^2}$ showing that it is even by the rule of even powers and also that the domain is balanced, being $[-4, 4]$.

5. (a) $(f \circ g)(x) = |x^2 - 3|$ (b) $(g \circ f)(x) = x^2 - 6x + 9$ (c) $f^{-1}(x) = x^2$, defined only for nonnegative $x$.

   Domains of $f$ and $g \circ f$: all real numbers $x$ such that $x \geq 0$

   Domains of $g$ and $f \circ g$: all real numbers

   The domain of $f^{-1}$ is $x \geq 0$.

   *Work for* $f \circ g$*:*

   $x^4 - 6x^2 + 9 = (x^2-3)^2$. Then $\sqrt{(x^2-3)^2}$ must have a positive answer.(It's the *positive* square root, after all.) That is accomplished by taking the absolute value of $x^2 - 3$. Writing the answer as $x^2 - 3$ is wrong. It is possible for $x$ to be positive and still have $x^2 - 3$ turn out to be negative. For example, consider what happens when $x = 1$: $g(1) = 4$, $f(4) = 2$. But if you had said that the answer was $x^2 - 3$, then $(f \circ g)(1) = 1^2 - 3 = -2$, and that is incorrect.

   The domain of $f \circ g$ is all reals because $x^2 - 6x^2 + 9$, being a perfect square, is always greater than or equal to zero—no matter what the value of $x$.

   *Work for* $f^{-1}$*:*

   Solve $x = \sqrt{y}$ for $y$ to get $y = x^2$, but note that $x$, being the nonnegative square root of $y$ must be greater than or equal to 0.

6. $1-x$, $\frac{1}{x}$, $\frac{x}{2x+1}$, $\frac{2x-1}{1-x}$, $\frac{x}{1-x}$. The domains are

   for $f$: all reals except $-1$; but for $g \circ f$: all reals except $-1$ and 0.

   for $g$ and $f \circ g$: all reals except 0.

   for $f \circ f$: all reals except $-1$ and $-1/2$.

   for $g \circ g$: all reals except 0 and 1.

   for $f^{-1}$: all reals except 1. Note that the inverse of $f$ turned out to be equivalent to the reciprocal of $g$.

7. $f^{-1}(x) = \dfrac{3x-4}{5x}$. It was found by solving $x = \dfrac{4}{-5y+3}$ for $y$.

   As a verbal string it is divide by 4; take reciprocal; subtract 3; change sign; divide by 5. This is the formula
   $$-\frac{1}{5}\left(\frac{4}{x} - 3\right).$$

   The two answers, while different formulas, are algebraically equivalent.

   The orginal verbal string was (before inversion) multiply by 5, change sign, add 3, take reciprocal, multiply by 4. All that had to be done was to find the inverse of each operation and apply them in the opposite order as they were in the original function.

   Because this problem is not easily conceptualized, one should apply the two results to a number of his choosing to see if they work.

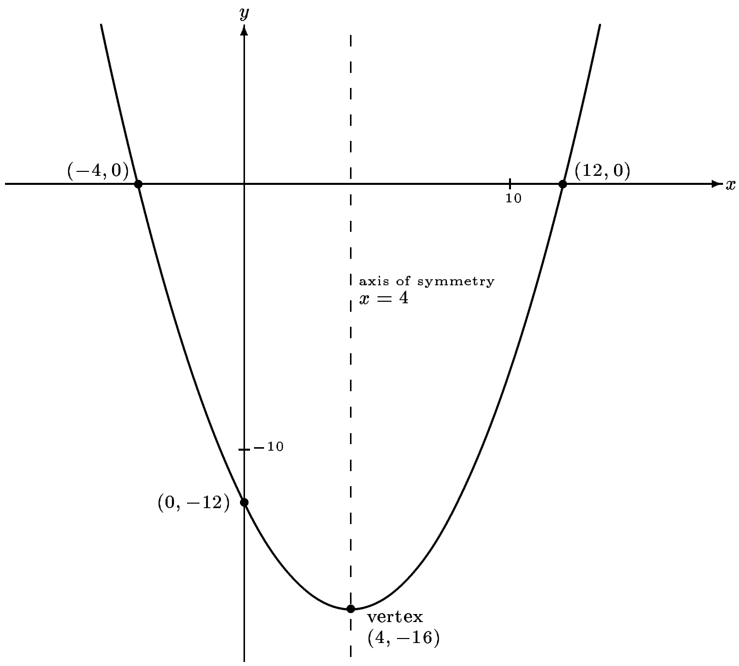
   Let $x = 3$, so $f(3) = \frac{4}{-15+3} = \frac{4}{-12} = -\frac{1}{3}$.

   Then $f^{-1}(-1/3) = \frac{3(-1/3)-4}{5(-1/3)} = \frac{-5}{-5/3} = 3$. It checks.

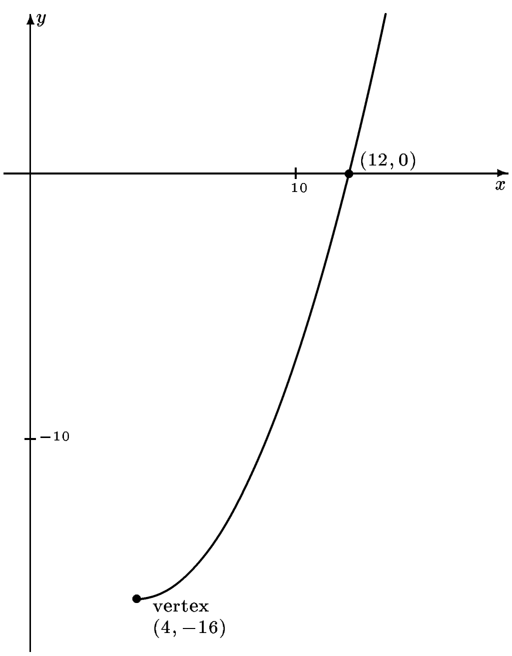
   And the verbal: $-1/3$ divided by 4 is $-1/12$, reciprocal is $-12$, subtract 3 gives $-15$, change sign gives 15, divide by 5 gives 3, as expected.

8. True.

9. $f(x) = \frac{1}{4}(x-4)^2 - 16.$




10. (a) $f(x) = \frac{1}{4}(x-4)^2 - 16.$

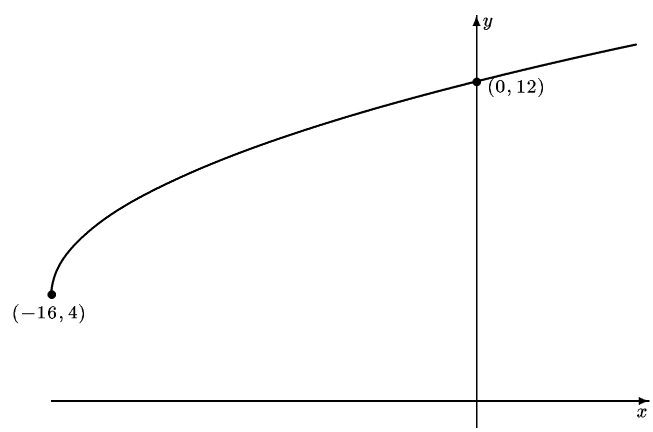



This equation describes $y$ as a function of $x$.

(b) It is one-to-one, so it has an inverse function.

The equation of the inverse function is $y = 2\sqrt{x+16} + 4$ for $x \geq -16$.

Switch variables and solve, using the standard form of the equation, from the previous problem.




11. Let $x =$ the width and let $z =$ the length. The area $= xz$.
Since $x + z = 26$, we have $z = 26 - x$. Substituting $26 - x$ for $z$, we find that the area $= xz = x(26-x) = 26x - x^2 = -x^2 + 26x$. So for the area we have $f(x) = -x^2 + 26x$ with $a = -1 < 0$. That means that the parbola opens down and has a maximum value of $f(x)$.

$$k = c - \frac{b^2}{4a} = 0 - \frac{26^2}{4(-1)} = -\frac{576}{-4} = \frac{-576}{-4} = 169 \text{ square feet}$$

The maximum area is equal to the maximum value of $f(x)$, which equals the maximum value of $y$, which is called $k$. Remember that the quantity to be maximized is represented by $y$.
This problem could also be done by completing the square the long way.
We find that $f(x) = -x^2 + 26x = -(x^2 - 26x) = -(x^2 - 26x + 169 - 169) = -(x-13)^2 + 169$.
Thus $f(x) = -(x-13)^2 + 169$ and the vertex is $(13, 169) = (h, k)$.

12. Call the two numbers $x$ and $w$. It is best to avoid using the variable $y$ because of the potential confusion with the actual output.

    Letting $x$ represent the first number, the result is $r = 3x^2 + 2w$. This problem can be reduced to two variables by using the conditions relating $w$ and $x$ linearly.

    $x + w = 20$.

    Solve for $w$ to get $w = 20 - x$.

    Substitute in the original equation to get $r = 3x^2 + 2(20 - x)$.

    $r = 3x^2 - 2x + 40$ is a quadratic function in $x$. It opens up and has a minimum.

    The minimum value of this function occurs when the first number is 1/3. It can be obtained from the vertex formula. It occurs when the 1st number $x = 1/3$, because by the vertex formula

    $$h = -b/2a = -(-2)/6 = 2/6 = 1/3.$$

    Use $w = 20 - x$ to determine that the second number is $20 - (1/3) = 19\frac{2}{3}$.

    Although not asked, the smallest possible result is $39\frac{2}{3}$.

    Had the variables been reversed, with $w$ the first number, $r = 3(20 - x)^2 + 2x = 3x^2 - 118x + 1200$, and the minimum value of $x$ would have been

    $$h = -b/2a = -(-118)/6 = 59/3 = 19\frac{2}{3}.$$

    That is now the second number based on the reassigned variables.

    Then the first number is $20 - 19\frac{2}{3} = 1/3$.

    The smallest possible result is still $39\frac{2}{3}$.

    *Answer:* The first number equals 1/3; and the second number equals $19\frac{2}{3}$.

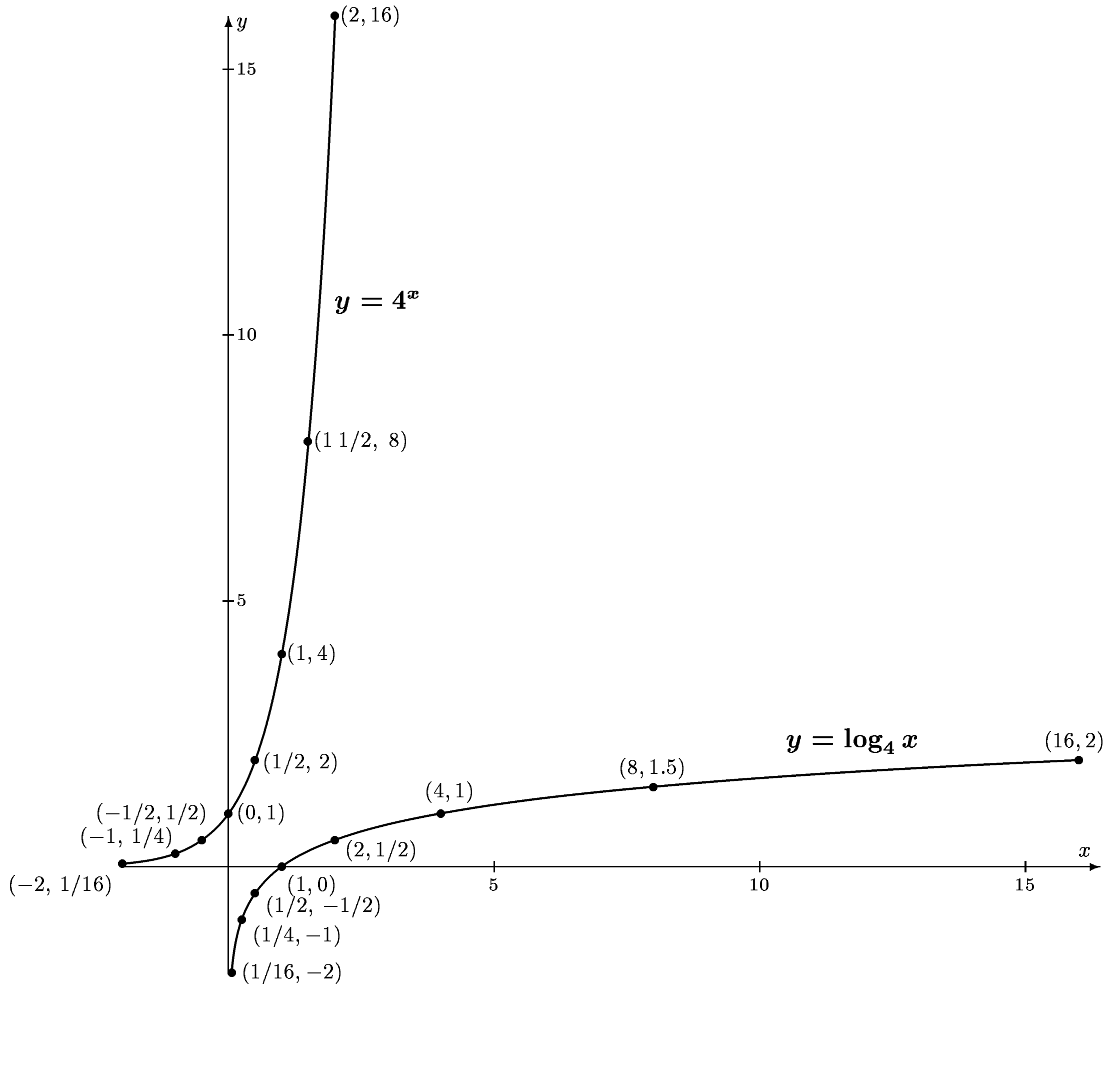
13. (a) For $f$; $(-2, 1/16)$, $(-1, 1/4)$, $(-1/2, 1/2)$, $(0, 1)$, $(1/2, 2)$, $(1, 4)$, $(1\frac{1}{2}, 8)$, $(2, 16)$.

For $g$: $(\frac{1}{16}, -2)$, $(\frac{1}{4}, -1)$, $(\frac{1}{2}, -1/2)$, $(1, 0)$, $(2, /, 1/2)$, $(4, 1)$, $(8, 1\frac{1}{2})$, $(16, 2)$.

The same numbers appear in each list, just that the order of each pair has been reversed.

When the $x$ and the $y$ coordinates are switched on a point, it reflects the point acrossw the line $y = x$. Either set of points has been reflected across the line $y = x$ to get the other set of points.

(b) These two graphs have the same shape. One has been reflected across the line $y = x$ to get the other. If the log were rotated 90 degrees counterclockwise and then flipped across the $y$-axis, it would land exactly on the graph of $4^x$. Try it.
The asymptote for the log graph is the negative $y$ axis, and the asymptote for $y = 4^x$ is the negative $x$-axis.




14. (a) $4w$ (b) $2w + 2$ (c) $2w + 4$ (d) $w/4$ (e) $w^2 + 4w + 4$ (f) $\sqrt{2w}$

15. (a) 2.838 (b) −0.8095 (c) 7.6485 (d) 0.87 (e) 4.7455 (f) −6.392 (g) 4.5495 (h) 0.69933

16. (a) $x = 20$ (b) $x = 500/33$ or $x = 15\frac{5}{33}$ (c) $x = 7$ (d) $x = 25$

17. Any solution must be a positive number for which $1 - 3x > 0$ also. That means $x < 1/3$, and we have $0 < x < 1/3$. A valid solution must lie in the interval $(0, 1/3)$.

    By the product rule for logarithms: $\log_2(x - 3x^2) = -4$.

    Rewriting in exponent form: $2^{-4} = x - 3x^2$.

    Then $3x^2 - x + \frac{1}{16} = 0$.

    This could be solved by the quadratic formula, or by multiplying out by 16 then factoring, or by straight factoring using fractions.

    Quadratic formula: $\dfrac{1 \pm \sqrt{1 - 4(3)(1/16)}}{6} = \dfrac{1 \pm \frac{1}{2}}{6}$.

    The answers are $\frac{3/2}{6} = \frac{3}{12} = 1/4$ and $\frac{1/2}{6} = 1/12$.

    Multiply out: $48x^2 - 16x + 1 = 0$, factor as $(12x - 1)(4x - 1) = 0$, so $12x - 1 = 0$ and $x = 1/12$, or $4x - 1 = 0$ and $x = 1/4$.

    Simple factoring with fractions: $(3x - 1/4)(x - 1/4) = 0$ will give $3x - 1/4 = 0$, $x = 1/12$, and $x - 1/4 = 0$, $x = 1/4$.

    Check when $x = 1/4$:

    $\log_2(1/4) + \log_2(1 - 3(1/4)) = -2 + \log_2(1/4) = -2 + (-2) = -4$; it's OK.

18. Any solution must be a positive number for which $1 - 2x > 0$ also. That means $x < 1/2$, and we have $0 < x < 1/2$. A valid solution must lie in the interval $(0, 1/2)$.

    By the product rule for logarithms: $\log_2(x - 2x^2) = -3$.

    Rewriting in exponent form: $2^{-3} = x - 2x^2$.

    Then $2x^2 - x + \frac{1}{8} = 0$.

    This could be solved by the quadratic formula, or by multiplying out by 8 then factoring, or by straight factoring using fractions.

    Quadratic formula: $\dfrac{1 \pm \sqrt{1 - 4(2)(1/8)}}{4} = \dfrac{1 \pm 0}{4} = 1/4$.

    Multiply out: $16x^2 - 8x + 1 = 0$, perfect square $(4x - 1)^2 = 0$, so $4x - 1 = 0$ and $x = 1/4$.

    Simple factoring with fractions: $(2x - 1/2)(x - 1/4) = 0$ will give $2x - 1/2 = 0$, $x = 1/4$, and $x - 1/4 = 0$, $x = 1/4$.

    Check:

    $\log_2(1/4) + \log_2(1 - 2(1/4)) = -2 + \log_2(1/2) = -2 + (-1) = -3$; it's OK.

19. All solutions must be negative.

    The solution is $x = -1/6$, which is negative.

    Work: $\log_2\left(\frac{-x}{1-2x}\right) == 3$.

    $\frac{-x}{1-2x} = 2^{-3} = 1/8$.

    $-8x = 1 - 2x$.

    $6x = -1$, so $x = -1/6$.

    Check: $\log_2(-(-1/6)) - \log_3(1 - 2(-1/6)) = -3$.

    $\log_2(1/6) - \log_2(8/6) = -3$.

    $\log_2(\frac{1/6}{8/6}) = \log_2(1/8) = -3$, so it checks.

20. (a) True.

    (b) True. Be careful. The denominator is the log of the square root, not the square root of the log.

    (c) False. It is not true when $a$ is negative. But change the right side to $2\log|a| - 2$ and it is true.

    (d) True.

21. False. The base of a logarithm must be positive.

22. (a) True.
    $\log_a uv = \log_a u + \log_a v$
    This is the first property of logarithms.
    (b) False.
    $(\log_a u) \div (\log_a v) = \log_a(u - v)$
    $2 = \log 100 \div \log 10 \neq \log(100 - 10) = \log 90 \approx 1.95$
    (c) True.
    $(\log_a b)(\log_b a) = (\log_a b)\left(\frac{\log_a a}{\log_a b}\right) = \log_a a = 1$. The key was changing the base of $\log_b a$ to base $a$.

23. (a) False. Try it with $a = 2$, $b = 4$, and $c = 8$.
    (b) True. Take $\log_b$ of each and show that the results are the same.

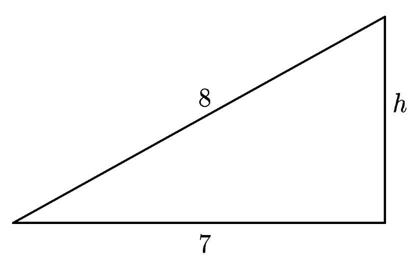
24. About 566 bacteria. The one-hour growth factor is $\sqrt[4]{4} = \sqrt{2}$, and $400\sqrt{2} \approx 566$.

25. (a) 7.5 hours.
    (b) False. There were exactly 8 pounds of frut flies at time 17 hours, not at time 8 hours.
    (c) $2^{1.8} \approx 3.4822$ pounds of fruit flies.
    (d) Exactly at time 39 hours and 30 minutes.

26. Convert 80° to radians: $80° \times \frac{\pi \text{ radians}}{180°} = \frac{4\pi}{9}$.
    80 feet $= \frac{4\pi}{9} \times r$. Since $\frac{4\pi}{9}$ is a unitless ratio, do not include the word radians in this equation.
    $r = \frac{180}{\pi}$ feet, after solving.

27. $\sin\theta = \dfrac{\sqrt{15}}{8} \quad \cos\theta = \dfrac{7}{8} \quad \tan\theta = \dfrac{\sqrt{15}}{7} \quad \csc\theta = \dfrac{8\sqrt{15}}{15} \quad \cot\theta = \dfrac{7\sqrt{15}}{15}$

    $\sin 2\theta = \frac{7}{32}\sqrt{15} \quad \cos 2\theta = 17/32$




    To find the third side, use $h^2 + 7^2 = 8^2$ so $h = \sqrt{64 - 49} = \sqrt{15}$.
    Then $\sec(90° - \theta) = 1/\cos(90° - \theta) = 1/\sin\theta = \csc\theta = \frac{8\sqrt{15}}{15}$.
    And $\csc^2\theta - 1 = \cot^2\theta = \left(\frac{7}{\sqrt{15}}\right)^2 = 49/15$.

28. (a) $\dfrac{1}{1 - 2r^2}$.
    (b) $\frac{\sqrt{2}}{2}(\sqrt{1 - r^2} + r)$.
    (c) $\pi - \arcsin r$.

29. (a) $\dfrac{r}{\sqrt{1 - r^2}}$.
    (b) $1 - r^2$.

30. The sine of $\theta$ is positive and is equal to $\sqrt{1-\left(\frac{7}{25}\right)^2}$, which comes out to be $\frac{24}{25}$.

    Then $\cos\left(\theta+\frac{\pi}{4}\right)=\cos\theta\cos\frac{\pi}{4}-\sin\theta\sin\frac{\pi}{4}=\frac{7}{25}\cdot\frac{\sqrt{2}}{2}-\frac{24}{25}\cdot\frac{\sqrt{2}}{2}=\frac{-17\sqrt{2}}{50}=-0.34\sqrt{2}$.

    Calculator verification: $\theta$ was about 73.74 degrees, the arccosine of 0.28.
    The cosine of $73.74+45$ degrees is the cosine of 118.74 degrees, about $-0.4808$.
    But $-0.34\sqrt{2}\approx-0.34\times1.414213562=-0.4808$.
    It checks.

31. $\sin\left(\theta+\frac{\pi}{3}\right)=\sin\theta\cos\frac{\pi}{3}+\cos\theta\sin\frac{\pi}{3}=\frac{4}{5}\cdot\frac{1}{2}+\frac{3}{5}\cdot\frac{\sqrt{3}}{2}=\frac{4+3\sqrt{3}}{10}$. (Used $\sin^2\theta+\cos^2\theta=1$ to get $\sin\theta$.)

    Calculator verification: $\theta$ was about 53.1301 degrees, the arccosine of 0.6.
    Adding 60 dgrees, gives 113.1301 degrees, and the sine of that is about 0.9196.
    Evaluating $\frac{4+3\sqrt{3}}{10}$ gives one tenth of $4+3(1.73205)=9.19615$, which is also 0.9196. It checks.

32. The answers are: $\sin 3\theta=117/125$ and $\cos 3\theta=44/125$.
    Work: $\sin^2\theta+\cos^2\theta=1$, so $\sin^2\theta+16/25=1$, $\sin^2\theta=9/25$, $\sin\theta=\pm3/5$.
    With the angle (or arc) in the second quadrant, the sine will be positive, making $\sin\theta=3/5$.

    Double angles: $\sin 2\theta=2\sin\theta\cos\theta=2(3/5)(-4/5)=-24/25$.
    $\cos 2\theta=\cos^2\theta-\sin^2\theta=(-4/5)^2-(3/5)^2=16/25-9/25=7/25$.

    Triple angles: $\sin 3\theta=\sin(2\theta+\theta)=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta=(-24/25)(-4/5)+(7/25)(3/5)=$
    $96/125+21/125=117/125$.

$\cos 3\theta=\cos(2\theta+\theta)=\cos 2\theta\cos\theta-\sin 2\theta\sin\theta=(7/25)(-4/5)-(-24/25)(3/5)=(-28/125)+(72/125)=44/125$.

33. First find $\sin\theta$, needed for part (b). It will be negative, so it is $-\sqrt{1-(-4/5)^2}=-3/5$.

    (a) Because $\frac{1}{2}\theta$ is between $\frac{3\pi}{4}$ and $\pi$, $\cos\frac{1}{2}\theta$ will be negative.
    $\cos\frac{1}{2}\theta=-\sqrt{\frac{1+\cos\theta}{2}}=-\sqrt{\frac{1+4/5}{2}}=-\sqrt{\frac{9}{10}}=-3/\sqrt{10}=-\frac{3}{10}\sqrt{10}$.

    (b) The sine and cosine of $\pi/4$ are both equal to $\sqrt{2}/2$.
    $\cos(\theta+\frac{\pi}{4})=\cos\theta\cos\frac{\pi}{4}-\sin\theta\sin\frac{\pi}{4}=(4/5)(\sqrt{2}/2)-(-3/5)(\sqrt{2}/2)=\frac{7}{10}\sqrt{2}$.

34. $\cos\theta=+\sqrt{1-(24/25)^2}=\sqrt{(1-(576/625)}=\sqrt{(49/625)}=7/25$. (This arcsine is in the first quadrant.)

    $\sin\frac{1}{2}\theta=+\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{1-7/25}{2}}=\sqrt{\frac{9}{25}}=3/5$. (Positive because $\frac{1}{2}\theta$ is also in the first quadrant.)

    $\cos\frac{1}{2}\theta=+\sqrt{\frac{1+\cos\theta}{2}}=\sqrt{\frac{1+7/25}{2}}=\sqrt{\frac{16}{25}}=4/5$.

35. $\sin\frac{1}{2}\theta=+\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{1-1/8}{2}}=\sqrt{\frac{7}{16}}=\sqrt{7}/4$, because $\theta$ is in the first quadrant and so is $\frac{1}{2}\theta$.

    $\cos\frac{1}{2}\theta=+\sqrt{\frac{1+\cos\theta}{2}}=\sqrt{\frac{1+1/8}{2}}=\sqrt{\frac{9}{16}}=3/4$.

    To check the work, estimate the angles on a calculator: $\theta\approx82.819244°$ and $\frac{1}{2}\theta\approx41.409622$.
    The cosine comes out to 0.75, almost exactly, and the sine comes out to 0.6614378, which is about $\sqrt{7}/4$.

36. $\sin\frac{1}{2}\theta=+\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{1-7/9}{2}}=\sqrt{\frac{1}{9}}=1/3$, positive because $\frac{1}{2}\theta$ lies in the first quadrant.

    $\cos\frac{1}{2}\theta=+\sqrt{\frac{1+\cos\theta}{2}}=\sqrt{\frac{1+7/9}{2}}=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}=\frac{2}{3}\sqrt{2}$.

37. (a) First note that the half-angle is between $\pi/2$ and $3\pi/4$, so its cosine is negative.
    $\cos\frac{1}{2}\theta=-\sqrt{\frac{1+\cos\theta}{2}}=-\sqrt{\frac{1-7/9}{2}}=-\sqrt{\frac{1}{9}}=-1/3$.

    (b) The half-angle is between $\pi/2$ and $3\pi/4$, so its sine is positive.

38. First note that half-angle is between $\pi/2$ and $3\pi/4$, so its cosine is negative.

    $\cos\frac{1}{2}\theta=-\sqrt{\frac{1+\cos\theta}{2}}=-\sqrt{\frac{1+(-0.82)}{2}}=-\sqrt{\frac{0.18}{2}}=-\sqrt{0.09}=-0.3$.

39. (a) First note that half-angle is between 0 and $\pi$, so its sine is positive
$\sin\frac{1}{2}\theta = +\sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-(-0.62)}{2}} = \sqrt{\frac{1.62}{2}} = \sqrt{0.81} = 0.9.$

(b) One solution is the inverse cosine of $-0.62$, which is about 128.3161345 degrees.
Another solution is minus 128.3161345 degrees, but it is not the interval. Adding 360 degrees gives the same terminal point and the result will be between 0 and 360 degrees. So the second answer is $360 - 128.3161345 = 231.6838655$ degrees.
Check both of these on your calculator, being sure to be in degree mode. Their cosines should come exactly as required, when the angles are expressed to that precision.

40. First find $\sin\alpha$. It will be positive, so it is $+\sqrt{1-(1/4)^2} = \sqrt{\frac{15}{16}} = \sqrt{15}/4$.

Then find $\sin\beta$. It will be positive, so it is $+\sqrt{1-(7/8)^2} = \sqrt{1-\frac{49}{64}} = \sqrt{\frac{15}{64}} = \sqrt{15}/8$.

(a) It looks like the sine of $\alpha$ is twice the sine of $\beta$. That's fine since $\alpha$ had the smaller cosine.

(b) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = (\sqrt{15}/4)(7/8) + (1/4)(\sqrt{15}/8) = \frac{7\sqrt{15}}{32} + \frac{\sqrt{15}}{32} = \frac{8\sqrt{15}}{32} = \sqrt{15}/4.$
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = (1/4)(7/8) - (\sqrt{15}/4)(\sqrt{15}/8) = \frac{7}{32} - \frac{15}{32} = -\frac{8}{32} = -1/4.$

(c) Since $\theta$ is the sum of two positive angles each less than $\pi/2$, it must be between 0 and $\pi$. Then half of $\theta$ must be between 0 and $\pi/2$. The sine of the half-angle will be positive in this case.
$\sin\frac{1}{2}\theta = +\sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-(-1/4)}{2}} = \sqrt{\frac{5}{8}} = \sqrt{\frac{10}{16}} = \sqrt{10}/4.$

(d) $\cos 2\theta = 2\cos^2\theta - 1 = 2(-1/4)^2 - 1 = 2(1/16) - 1 = 1/8 - 1 = -7/8.$
$\cos 4\theta = 2\cos^2 2\theta - 1 = 2(-7/8)^2 - 1 = 2(49/64) - 1 = 49/32 - 1 = 17/32.$

(e) These two angles have the same sine but are surely not equal, so they are supplementary: $\alpha + \theta = \pi$.
As expected, the cosines are negatives of each other, confirming that the two angles are supplementary.

41. 1.000000.

One such small number is 0.000000037.

If you got 0.017453, the calculator was incorrectly set to degree mode. Unless otherwise specified, one should assume that $s$ is in radian measure.

42. (a) $\cos x(1+\tan x)(1-\tan x) = \cos x(1-\tan^2 x) = \cos x(1-(\sec^2 x - 1)) = \cos x(2-\sec^2 x) =$

$$2\cos x - \frac{1}{\cos x} = \frac{2\cos^2 x - 1}{\cos x}.$$ This answer could also be given as $\frac{\cos 2x}{\cos x}$.

(b) $\tan x\cos^2 x = (\sin x/\cos x)\cos^2 x = \sin x\cos x = \frac{1}{2}\sin 2x.$

(c) $\cos^4 x - \sin^4 x = (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) = (1)(\cos 2x) = \cos 2x$

(d) $\frac{1+\cot^2 x}{\sin x} = \frac{\csc^2 x}{\sin x} = \csc^2 x\left(\frac{1}{\sin x}\right) = \csc^2 x(\csc x) = \csc^3 x.$

(e) $\frac{\sec x}{\csc x} = \left(\frac{1}{\cos x}\right)\left(\frac{1}{\csc x}\right) = \left(\frac{1}{\cos x}\right)\sin x = \frac{\sin x}{\cos x} = \tan x.$

(f) $\frac{\sec x}{\sin x} = \sec x\left(\frac{1}{\sin x}\right) = \left(\frac{1}{\cos x}\right)\left(\frac{1}{\sin x}\right) = \frac{1}{\sin x\cos x} = \frac{1}{(1/2)\sin 2x} = 2\csc 2x.$

43. $(2\cos^2\theta - 1) - 2\cos^2\theta = -1.$

or

$(1-2\sin^2\theta) - 2(1-\sin^2\theta) = 1 - 2\sin^2\theta - 2 + 2\sin^2\theta = -1.$

Check it by assigning a value to $\theta$, say $\pi/3$, and evaluating.

$\cos 2\pi/3 = -1/2$ and $\cos\pi/3 = 1/2$, so the expression ends up as

$-1/2 - 2(1/2)^2 = -1/2 - 2/4 = -1$, and it checks.

44. Start with $\sin 2x = 2\sin x\cos x$ and replace $x$ with $\frac{1}{2}\theta$. Finally divide both sides of the equation by 2.

Answer: $\frac{1}{2}\sin\theta$.

45. Assume that $\dfrac{\sin\frac{1}{2}\theta}{\frac{1}{2}\sin\theta}$ is equal to $\dfrac{1}{\cos\frac{1}{2}\theta}$.

Multiply it out to get $\cos\frac{1}{2}\theta \cdot \sin\frac{1}{2}\theta = \frac{1}{2}\sin\theta$ and $2\sin\frac{1}{2}\theta \cdot \cos\frac{1}{2}\theta = \sin\theta$.

That will be just an application of the double-angle formula for the sine using $\frac{1}{2}\theta$ for $x$. Integer multiples of $\pi$ will make the denominator of the original fraction equal to zero, so they must be excluded.

46. An equivalent equation in terms of $\cos x$ is obtained by substituting $2\cos^2 x - 1$ for $\cos 2x$.
$3(2\cos^2 x - 1) = 2\cos^2 x$, so $4\cos^2 = 3$, $\cos^2 x = 3/4$, and $\cos x = \pm\sqrt{3}/2$.

For $+\sqrt{3}/2$ the answers are $\pi/6$ and $-\pi/6$, which becomes $11\pi/6$ in the interval specified.
For $-\pi/6$ the primary solution is $5\pi/6$, the supplement of $\pi/6$. The other solution is $-5\pi/6$, which becomes $7\pi/6$ in the interval specified.

Answers: $\pi/6$, $5\pi/6$, $7\pi/6$, and $11\pi/6$.

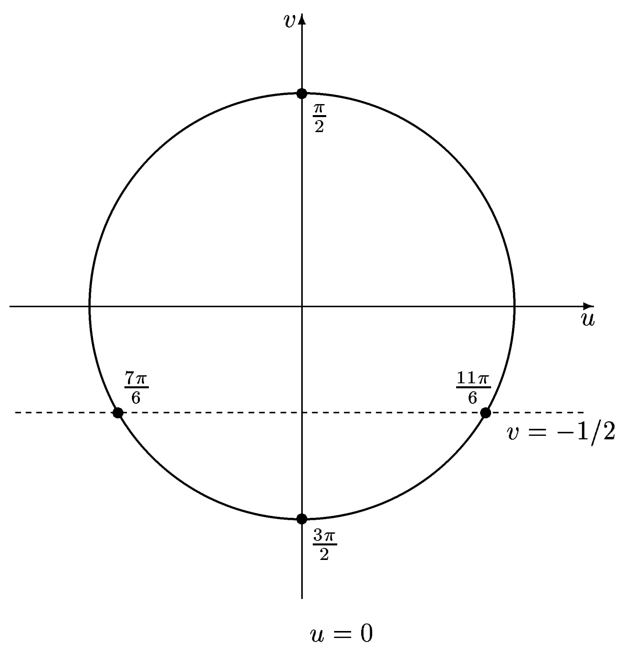
47. (a)

$$2\sin x\cos x = -\cos x$$
$$2\sin x\cos x + \cos x = 0$$
$$\cos x(2\sin x + 1) = 0$$

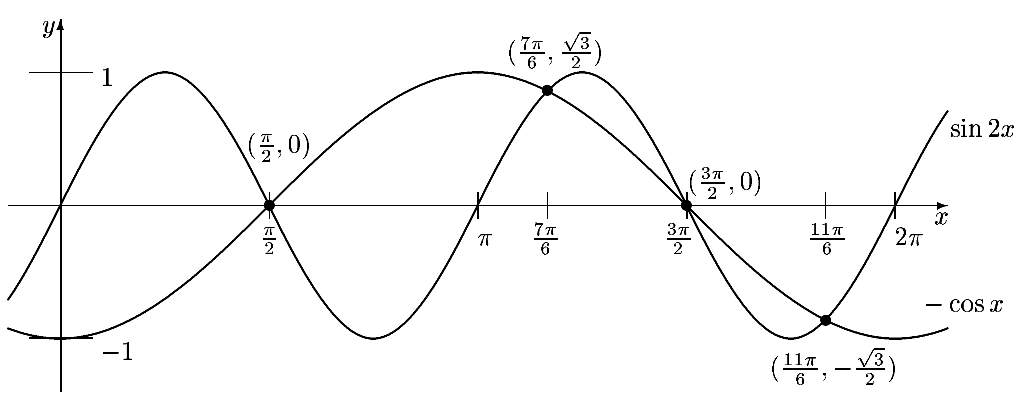
$\cos x = 0$ or $2\sin x + 1 = 0$

$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$

$2\sin x = -1$
$\sin x = -1/2$
$x = \frac{7\pi}{6}$ or $x = \frac{11\pi}{6}$




(b)



48. (a)

$$2\sin x\cos x = -\sin x$$
$$2\sin x\cos x + \sin x = 0$$
$$\sin x(2\cos x + 1) = 0$$

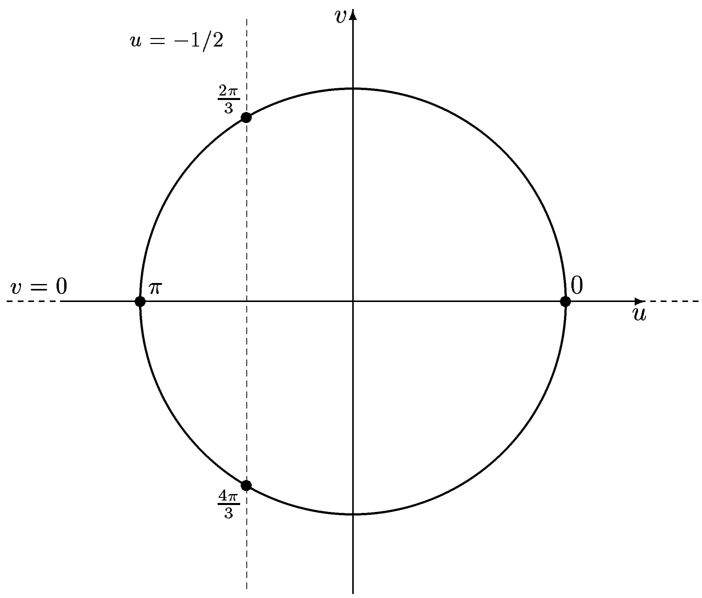
$\sin x = 0$ or $2\cos x + 1 = 0$

$x = 0$ or $x = \pi$ or $x = 2\pi$

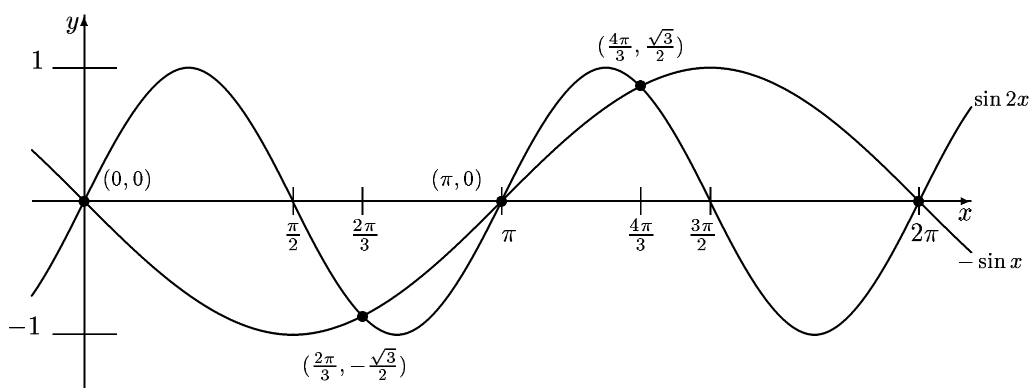
$2\cos x = -1$

$\cos x = -1/2$

$x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$




(b)



49. The third side is 1 long, the angle opposite the side with length $\sqrt{3}$ is 120°, and the third angle must be 30° because the triangle is isosceles. Note that these 3 angles add to 180°.

    Work: $c^2 = 1^2 + (\sqrt{3})^2 - 2(1)(\sqrt{3})\cos 30° = 1 + 3 - 2\sqrt{3}(\frac{\sqrt{3}}{2}) = 1 + 3 - 3 = 1$, so $c = 1$.

    Then use the fact that this triangle is isosceles to get the other two angles immediately.

50. The third side is 7 long, the angle opposite the side with length $5\sqrt{3}$ is 141.787°, and the angle opposite the side with length 2 is 8.213°. Note that these 3 angles add to 180.000°.

    The longest side is $5\sqrt{3}$, so the largest angle will be between the sides of lengths 2 and 7.

    The shortest side is 2, so the smallest angle will between the sides of lengths 7 and $5\sqrt{3}$.

    Work: $c^2 = 2^2 + (5\sqrt{3})^2 - 2(2)(5\sqrt{3})\cos 30° = 4 + 75 - 20\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 79 - 30 = 49$, so $c = 7$.

    Largest angle: $\cos^2 = \frac{2^2+7^2-(5\sqrt{3})^2}{2\cdot 2\cdot 7} = \frac{53-75}{28} = \frac{-22}{28} = \frac{-11}{14}$.

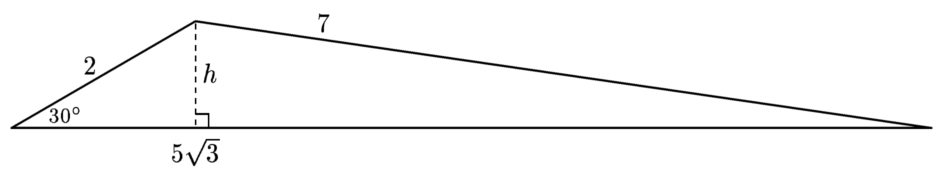
    Then the angle is the inverse cosine of $-11/14$, which comes out to about 141.787° on a calculator.

    Best practice is to find the third angle by the law of cosines, then check that all three add up to 180°. This will show if errors were made.

    Smallest angle: $\cos^2 c = \frac{7^2+(5\sqrt{3})^2-2^2}{2(7)(5\sqrt{3})} = \frac{49+75-4}{70\sqrt{3}} = 120/(70\sqrt{3})$.

    Then the angle is the inverse cosine of that expression, which comes out to about 8.213° on a calculator.

    The three angles add up to exactly 180.000°.




For the area:

$\frac{h}{2} = \sin 30° = \frac{1}{2}$.

So $h = 1$.

The area $= \frac{1}{2}bh = \frac{1}{2}\cdot 5\sqrt{3}\cdot 1 = 2.5\sqrt{3}$.

51.

The period is $\frac{2\pi}{|a|} = \frac{2\pi}{2} = \pi$.
The amplitude is 5.

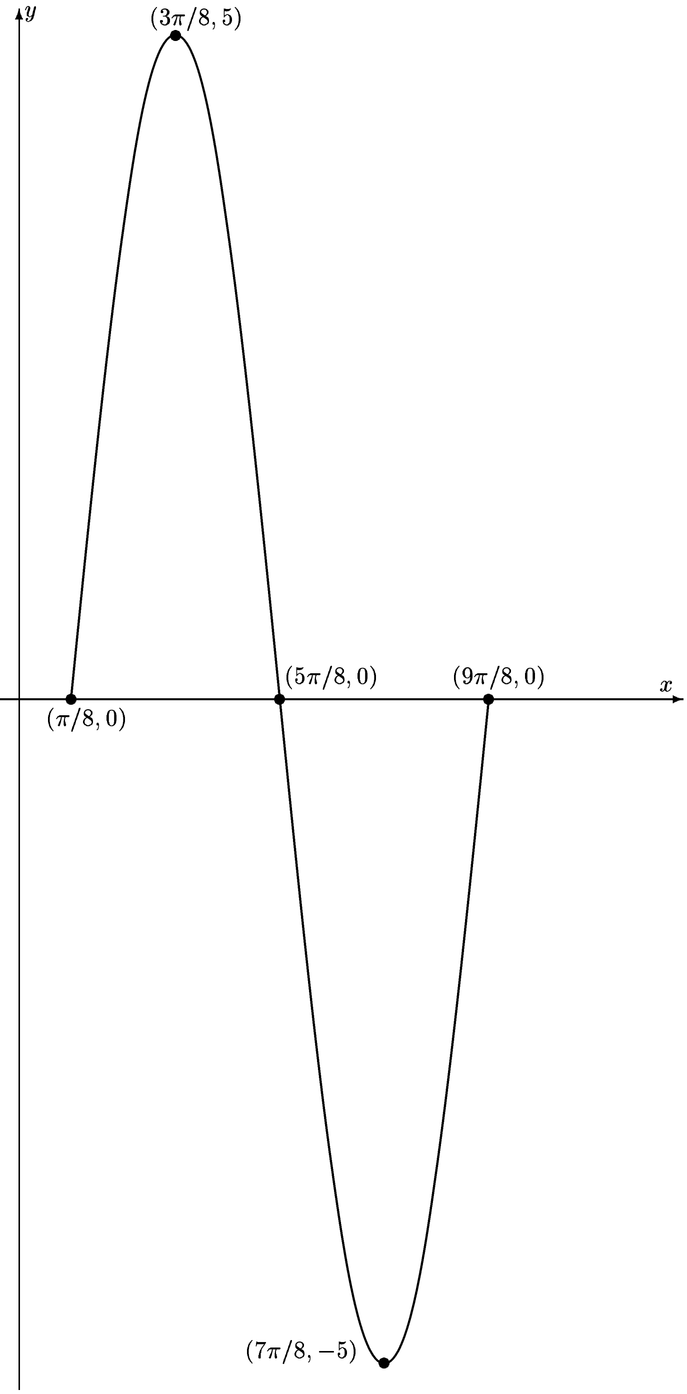
The phase fraction is $\frac{-(-\pi/4)}{2\pi} = 1/8$.

Factor as $5\sin\left[2\left(x - \frac{\pi}{8}\right)\right]$.

So shift the graph of $y = 5\sin 2x$ $\pi/8$ to the right.

Before the shift, $5\sin 2x$ had $x$-intercepts at $0$, $\pi/2$, and $\pi$, and highest and lowest points at $x = \pi/4$ and $x = 3\pi/4$.

Adding $\pi/8$ to each of those five gives the pictured result.




52. (a) $-11/16$

(b) $\sqrt{2}/8 - \sqrt{30}/8$

(c) $\frac{1}{4}\sqrt{6}$ and $\frac{1}{4}\sqrt{10}$.