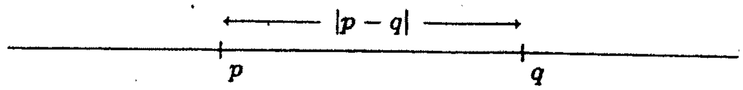
Equations and Inequalities in $|x \pm a|$.

When we graph numbers on a number-line, a basic rule states that: for any two numbers p and q,

$$|p - q| = \text{the distance between the points marked } p \text{ and } q.$$

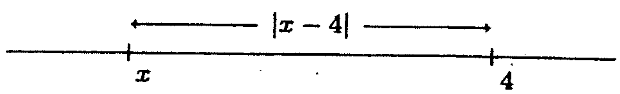


This fact can be used to solve equations and inequalities in $|x \pm a|$.

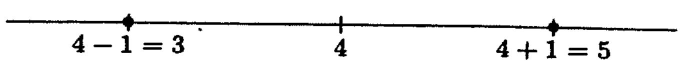
- To solve an equation or inequality in $|x - a|$, think of $|x - a|$ as the distance from the *variable* point x to the *central* point a. Then find all x for which the distance to the center, a, satisfies the given conditions on $|x - a|$.

Example 1. Solve the equation $1 = |x - 4|$.

If we plot x and 4 on a number-line, then $|x - 4|$ is the distance between those points.



The equation requires that this distance be 1: any solution x is plotted as a point of distance 1 from the "center" 4. There are two such points, which we find by moving one unit to either side of 4.

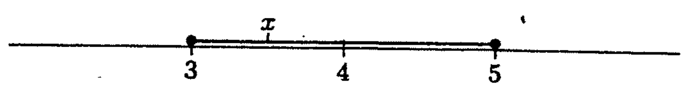


So the solutions to $1 = |x - 4|$ are, $x = 3$ and $x = 5$.

Check: $(x = 3)$ $|3 - 4| = |-1| = 1$; $(x = 5)$ $|5 - 4| = |1| = 1$.

Example 2. Solve the inequality $1 \geq |x - 4|$.

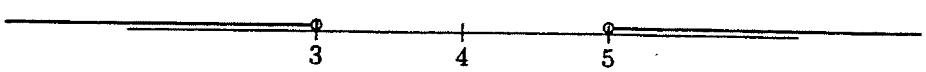
This inequality requires that the distance between the variable point x and the center 4 be 1 or less. We just found that, when $x = 3$ or $x = 5$, the distance is exactly 1. Any point *in between* 3 and 5 will be closer to 4. The set of such points is the interval with ends marked 3 and 5, and



center 4, including the ends. So the solution to $1 \geq |x - 4|$ is the set $\{x \mid 3 \leq x \leq 5\}$ of numbers at least 3 but not more than 5. For instance, $x = 3.5$ is a solution, since $|3.5 - 4| = |-0.5| < 1$.

Example 3. Solve the inequality $1 < |x - 4|$.

Now the distance between points marked x and 4 has to be greater than 1. Any point strictly to the left of 3 or to the right of 5 will satisfy the condition. The set of such points is a pair of intervals



as shown, with ends marked 3 and 5; the ends are not included. So the solution to $1 < |x - 4|$ is, the set of all numbers, either less than 3 or greater than 5:

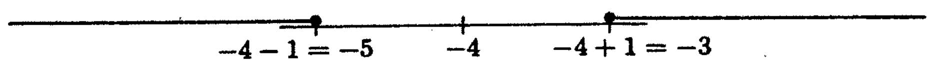
$$\{x \mid x < 3 \quad \text{or} \quad x > 5\}.$$

There is a minus sign in the distance rule, for it speaks of $|p-q|$. The examples so far have involved $|x-4|$, which fits the form $|p-q|$. But $|x+4|$, for instance, does not. However, we can make it fit by a simple adjustment: write $x+4=x-(-4)$; then an equation or inequality in $|x+4|$ can be solved by applying the distance rule to $|x-(-4)|$, regarded as the distance from x to the central point -4. In general we do the same:

- To solve an equation or inequality in $|x+a|$, rewrite $|x+a|$ as $|x-(-a)|$. This is the distance from the variable point x to the central point $-a$, so look for all x whose distance to $-a$ satisfies the given conditions on $|x+a|$.

Example 4. Solve the inequality $1 \le |x+4|$.

Write the inequality as $1 \le |x-(-4)|$. Then it requires that x be plotted as a point of distance at least 1 from the central point -4.



The set of such points is a pair of intervals as shown, with ends, marked -5 and -3, included. So the solution to $1 \le |x+4|$ is,

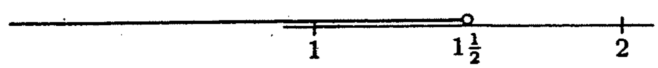
$$\{x \mid x \le -5 \quad \text{or} \quad x \ge -3|.$$

The last example involves two expressions in the form $|p-q|$.

Example 5. Solve the inequality $|x-1| < |x-2|$.

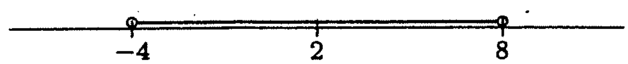
According to the distance rule, any solution x will be plotted as a point which is closer to the point 1 than to the point 2: the distance to 1 has to be less than the distance to 2. Now $1\frac{1}{2}$ is halfway between 1 and 2. Any point to the left of point $1\frac{1}{2}$ is closer to 1 than to 2. So the solution is,

$$\{x \mid x < 1\tfrac{1}{2}|.$$

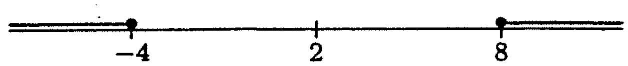


Exercises. Solve the equation or inequality, and graph the solution. Identify the "central point" in each case.

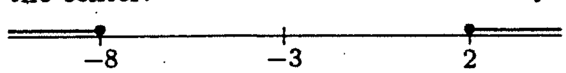
1. $|x-2|=6$ — *Ans.* The center is 2; moving 6 each way gives $x=-4$ or $x=8$.
2. $|x-2|<6$ — *Ans.*



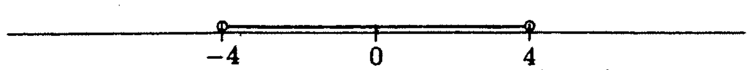
3. $|x-2| \ge 6$ — *Ans.*



4. $|x+3|=5$ — *Ans.* Write $|x+3|=|x-(-3)|$, to suit the rule. The center is then -3; moving 5 each way gives $x=-8$ or $x=2$.
5. $|x+3| \ge 5$ — *Ans.* The center is -3, and x has to be a distance *at least* 5 from the center.



6. $|x|=4$ — *Ans.* We can write $|x|=|x-0|$, so the center is 0. Moving 4 either side of 0 gives $x=-4$ or 4.
7. $|x|<4$ — *Ans.*



8. $|x+1|>|x-1|$ — *Ans.* x has to be closer to 1 than to -1, so the solution is the set $\{x \mid 0 < x|$.