Extreme Value Example

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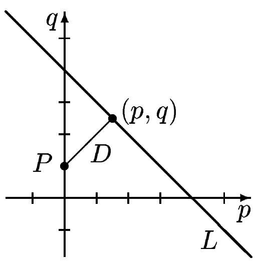


Let L be the line with equation $p + q = 4$ and let P be the point $(0, 1)$. Find the point on L closest to P, and find the distance between P and that point.

1. Assignment of variables:

   Let (p, q) be a typical point of L. Then

   $p =$ the first coordinate of a point on the line;

   $q =$ the second coordinate of a point on the line;

   $D =$ the distance between P and (p, q).

2. Formula for the quantity to be maximized:

   We seek the point (p, q) on L for which D has the least value. Now D is least whenever D^2 is least, and the latter is more convenient to use because it does not involve the square root of a complicated polynomial. We can express D^2 in terms of p and q by

   $$D^2 = p^2 + (q - 1)^2.$$

   (We used the distance formula with $p_1 = 0$, $p_2 = p$, $q_1 = 1$, and $q_2 = q$.)

3. Reduction to one variable, using an equation relating the two variables:

   (Although the relationship in this example is linear, in general it need not be.)

   When a point (p, q) is on L, its coordinates satisfy $p + q = 4$, the equation defining L. We can rewrite this as $q = 4 - p$ and substitute to get D^2 as a quadratic function of p alone.

   Now substitute $4 - p$ for q in the formula $D^2 = p^2 + (q - 1)^2$. Since the resulting equation expresses D^2 as a quadratic polynomial in p, we may look at D^2 as a function of p and use the notation $D^2 = f(p)$.

   $D^2 = p^2 + (q - 1)^2 = p^2 + \left((4 - p) - 1\right)^2 = p^2 + (3 - p)^2 = 2p^2 - 6p + 9$

   $D^2 = f(p) = 2p^2 - 6p + 9$ (a quadratic function of p)

   Since $a = 2 > 0$, the graph opens up and has a minimum point.

   We seek the p for which this quadratic function has the least value.

4. Finding the vertex by completing the square:

   $f(p) = 2p^2 - 6p + 9 = 2(p^2 - 3p) + 9 = 2\left(p^2 - 3p + (\frac{3}{2})^2 - (\frac{3}{2})^2\right) + 9 =$

   $2(p - \frac{3}{2})^2 - 2(\frac{9}{4}) + 9 = 2(p - \frac{3}{2})^2 - \frac{9}{2} + 9 = 2(p - \frac{3}{2})^2 + \frac{9}{2}$

   So $h = \frac{3}{2}$ and $k = \frac{9}{2}$. (The shortcut methods could also be used.)

   The vertex is $(\frac{3}{2}, \frac{9}{2})$.

   We have $D^2 = f(p) = \frac{9}{2}$ when $p = \frac{3}{2}$. That means that the square of the distance is $\frac{9}{2}$, a minimum, when the first coordinate of the point is $\frac{3}{2}$.

   Once we have completed the square, D^2 is thus expressed as a variable part, $2(p - \frac{3}{2})^2$, plus a fixed part, $\frac{9}{2}$, and D^2 is least when the variable part is zero since the variable part is always zero or larger. That is, D^2 is least when $p = \frac{3}{2}$, and then $D^2 = \frac{9}{2}$.

5. Conversion to solutions of the variables p, q, and D, assigned in the initial step:

   At the minimum point, the vertex, we have:
   $p = \frac{3}{2}$
   $q = 4 - p = 4 - \frac{3}{2} = \frac{5}{2}$.
   The point on the line L we require is $(\frac{3}{2}, \frac{5}{2})$.

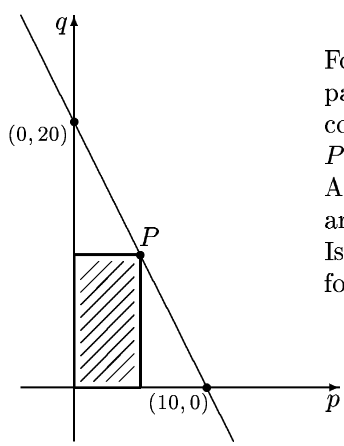
   The value k, the second coordinate of the vertex, which represents the minimum value of D^2, is equal to $\frac{9}{2}$.

   $D = \sqrt{D^2} = \sqrt{\frac{9}{2}} = 3/\sqrt{2}$

   The distance D between $(0, 1)$ and P, the closest point on the line, is $3/\sqrt{2} \approx 2.121$. To doublecheck, take the points $(0, 1)$ and $(\frac{3}{2}, \frac{5}{2})$ and apply the distance formula to them. The distance will be approximately 2.121, as previously found.

Exercises

In problems 1–3, for the given L and P, find the point on L closest to P, and find the distance between P and that point.

1. L: $2p + q = 3$ $\quad P = (1, 0)$ Ans. $(7/5, 1/5)$; $\sqrt{5}/5$
2. L: $p + 2q = 1$ $\quad P = (2, 2)$ Ans. $(1, 0)$; $\sqrt{5}$
3. L: $p - 2q = 1$ $\quad P = (0, 3)$ Ans. $(7/5, 1/5)$; $7/\sqrt{5}$
4. L: $3p - q = 8$ $\quad P = (0, 2)$ Ans. $(3, 1)$; $\sqrt{10}$
5. Find the point on the line $2p + q = 20$ closest to $(0, 0)$. Ans. $(8, 4)$
6. 

   For each point P on the line $2p + q = 20$, which passes through the points $(0, 20)$ and $(10, 0)$, consider the rectangle with upper right corner P and bounded by the axes, as in the sketch. Among these rectangles, find the one of largest area.
   Is the P of that rectangle the same point as was found in Problem 4?
   Ans. the rectangle with $P = (5, 10)$; no