Report by Stephen Parrott

on

Sufficient Conditions for uniqueness of the Weak Value

by

J. Dressel and A. N. Jordan

Note: This is version 2 of this report, submitted on October 30. After submitting the original version on October 27, I noticed some minor misstatements along with one substantial misunderstanding, which are corrected in this version. Sections 1 through 3 are identical to the original. The conclusions are substantially the same.

1 References

The following abbreviations for references are listed first, along with thumbnail sketches of their contents. The abbreviation DAJ will also be used to refer to the authors (Dressel, Ararwal, and Jordan) as a group, and DJ will refer to Dressel and Jordan.

DAJ: J. Dressel, S. Agarwal, and A. N. Jordan, "Contextual values of observables in quantum measurements", Phys. Rev. Lett. **104** 240401 (2010), arXiv:0911.4474v3

This four-page "Letter" introduces the authors' concept of "contextual values".

P1: S. Parrott, " 'Contextual weak values' of quantum measurements with positive measurement operators are not limited to the traditional weak value", arXiv:1102.4407, versions 1 through 6

A lengthy analysis of DAJ. Later versions differ substantially from earlier versions and include a counterexample to a major claim of DAJ, that "[their 'general conditioned average'] converges uniquely to the quantum weak value in the minimal disturbance limit".

P2: S. Parrott, "Introduction to 'contextual values' and a simpler counterexample to a claim of Dressel, Ararwal, and Jordan [Phys. Rev. Lett. **104** 240401 (2110)", arXiv:1105.4188, versions 1 through 6.

This actually contains two separate papers. The earlier versions presented a counterexample to the above claim of DAJ which is simpler than the one given in P1. Then Dressel and Jordan posted a reply in DJ1 (to be discussed below) claiming that the counterexample did not satisfy one of DAJ's hypotheses (the so-called "pseudo-inverse" prescription).

I do not agree with DJ1 that the pseudo-inverse prescription was presented in DAJ as a *hypothesis* for its claim about weak values, but DAJ does mention it in a vague way which leaves its significance unclear. Later versions of P1 include a different counterexample which *does* satisfy the "pseudo-inverse prescription", thus removing this as an issue.

The later counterexample was originally submitted to the arXiv as a separate paper which contained the introductory material of the earlier versions, but an arXiv moderator insisted that the later counterexample be appended to the earlier paper. To avoid a lengthy and unpleasant appeal, I did that. For this reason, the date of P2 (which is the unchangeable date of the first version) is prior to the date of DJ1 even though the later versions with the new counterexample were submitted in response to DJ1.

P2 also contains a critical analysis of various aspects of DJ1, including its attempt to justify the "pseudo-inverse prescription".

DJ1: J. Dressel and A. N. Jordan, "Sufficient conditions for uniqueness of the Weak Value", arXiv:1106.1871v1.

This is DJ's response to the earlier versions of P2. Though titled identically to the paper under review (subsequently called DJ3) and containing some of the same material, it differs substantially. I think DJ1 is easier to read than than DJ3.

DJ2: J. Dressel and A. N. Jordan, "Contextual Values Approach to the Generalized Measurement of Observables", arXiv:1110.0418v1

A lengthy expansion of DAJ. It contains the questionable claim of DAJ mentioned above as a "Theorem". It does not mention that this "Theorem" has been questioned in P1 and P2, which are not referenced at all.

DJ3: The paper under review, titled identically to DJ1 but substantially different. It questions the counterexamples of P2.

2 Disclaimer

I want to make clear that I am not a referee for DJ3. Therefore, I shall make no recommendation regarding its suitability for publication. I am commenting on it by request of the editors, in the hope that the comments may be helpful to them and the referees

3 General comments

I shall use the notation of DJ3, the submission under review, which is the same as the notation of DAJ and DJ1 (but not DJ2). It would be too time-consuming to separately define all of it here, so for the more technical comments I will assume that the reader is familiar with DJ3.

DJ3 covers much of the ground of DJ1, but seems to me harder to read. Indeed, I think it might be impossible to read carefully in a reasonable time by someone unfamiliar with DAJ and DJ1. Its main intent seems to be to invalidate the counterexamples of P2 to the claim of DAJ that

> "[its 'general conditioned average'] converges uniquely to the quantum weak value in the minimal disturbance limit."

Their objection to a counterexample in an early version of P2 is that it does not satisfy what they call the "pseudo-inverse prescription". This prescription requires that the contextual values $\vec{\alpha} = (\alpha_1, \ldots, \alpha_M)$, which *a priori* only need satisfy a "contextual values equation"

$$F\vec{\alpha} = \vec{a}$$

($\vec{a}$ being the eigenvalues of the "system observable" written as a vector, with F a given $N \times M$ matrix), be chosen as

$$\vec{\alpha} = F^+\vec{a}, \quad \text{where } F^+ \text{ denotes the Moore-Penrose pseudo-inverse of } F.$$

This is the "pseudo-inverse prescription". Since F need not be 1:1, there may be multiple solutions of the contextual value equation; the pseudo-inverse prescription singles out one of them.

The only reference to the pseudo-inverse prescription in DAJ is the single sentence[1]

> "... we propose that the physically sensible choice of CV [contextual values] is the least redundant set uniquely related to the eigenvalues through the Moore-Penrose pseudo-inverse."

No further explanation is given.

DJ1 and DJ3 do attempt to explain this, but unconvincingly in my opinion, for reasons given in detail in P2 which I shall not repeat here. However, I do want to make one further comment.

DJ1 and DJ3 try to relate the pseudo-inverse prescription to minimization of the "detector variance". They observe that the pseudo-inverse prescription minimizes a certain *upper bound* for the detector variance. (Of course, minimizing an upper bound for a quantity like detector variance does not guarantee minimizing the quantity itself.) DJ3 justifies the use of their particular upper bound as follows:

> "In absence of prior knowledge about the system one is dealing with, *this is the most general bound one can make* [emphasis mine]. Therefore, the pseudoinverse solution will choose the solution that provides the most rapid statistical convergence for observable measurements on the system given no prior knowledge of the system state."

[1] DAJ does devote a long paragraph to a complicated method of calculating the pseudo-inverse, but this has nothing to do with the reasons for using the pseudo-inverse in the first place.

The claim that "this is the most general bound one can make" is false, and they should know this because P2 proves a better bound. The proof is only a few lines and uses nothing more sophisticated than the Cauchy-Schwartz inequality. This really doesn't matter because the second sentence doesn't follow from the first, but it seems disturbing nevertheless.

Many questionable statements of DJ1 are repeated in DJ3. Since these have already been refuted (or so I believe) in P2 or P1, for brevity I shall not repeat the refutations here.

4 Are the counterexamples valid?

P2 gives two counterexamples to the original claim of DAJ (and repeated in all the DJ papers) that

> "[their 'general conditioned average'] converges uniquely to the quantum weak value in the minimal disturbance limit."

The earlier counterexample uses three measurement operators $\hat{\boldsymbol{M}}_{\boldsymbol{j}}$, $j = 1, 2, 3$ represented as 2×2 matrices. The contextual value equation has multiple solutions, and for a particular choice of solution, the resulting calculations become simple enough that all steps can be verified mentally. However, for that particular example, the contextual values which satisfy the pseudo-inverse prescription do not produce a counterexample.

The mathematics of this example is undisputed; the only issue is whether DAJ unequivocally requires the pseudo-inverse construction as a hypothesis for its "Weak values" section. Since the "Weak values" section of DAJ does not mention the pseudo-inverse prescription, and since the only justification for this prescription which DAJ or DJ have given is to attempt to minimize the detector variance (which has nothing to do with weak values), I consider that the example constitutes a counterexample to DAJ; DJ disagrees.

The second counterexample is unfortunately a bit more complicated, using three 3×3 measurement operators. However, it *does* satisfy the pseudo-inverse prescription because there is only one solution to the contextual value equation. So far as I know, its mathematics is undisputed. (However, as DJ3 helpfully points out, there *is* a typo which replaces a $\sqrt{1/3}$ in $\hat{\boldsymbol{M}}_2$ by an incorrect $1/3$, but the subsequent calculations use the correct value.)

DJ3's sole objection to the second counterexample claims that it does not satisfy another of the hypotheses for their "Theorem". To understand the objection, we must examine the relevant hypotheses in detail. They are (using the numbering of DJ3):

> "(iii) The equality $\hat{\boldsymbol{A}} = \sum_j \alpha_j(g)\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g)$ must be satisfied, where the contextual values $\alpha_j(g)$ are selected according to the pseudo-inverse prescription.
>
> (iv) The minimum nonzero order in g for all $\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g)$ is g^n such that (iii) is satisfied."

Before continuing, I invite the reader to try to understand precisely what (iv) means, particularly the phrase "such that (iii) is satisfied". Referees should already be familiar with the notation, but in case the reader is an editor who is not, the $\hat{\boldsymbol{E}}_{\boldsymbol{j}}$ are square matrices which depend analytically on a small real parameter g, $\hat{\boldsymbol{A}}$ is a constant matrix, and the contextual values $\alpha_j(g)$ are real-valued functions of g.

Here is how I interpreted it. "Minimum nonzero order in g" is not a standard mathematical phrase, but I suppose that the "minimum nonzero order in g for a power series

$$f(g) = f_0 + g^n \sum_{k=0}^{\infty} f_{n+k} g^k, \quad n > 0, \quad f_j \text{ constant for all } j,\ f_n \neq 0$$

would be n. It seems a bit strange (and quite restrictive) to assume that *all* of the $\hat{\boldsymbol{E}}_j(g) = \hat{\boldsymbol{E}}_j^{(0)} + g^n \sum_{k=0}^{\infty} \hat{\boldsymbol{E}}_j^{(k+n)}$ should have the *same* minimum nonzero order n, but let us pass over this point. For expositional simplicity in the rest of the report, I will always assume that $n = 1$.

I puzzled over the last phrase in (iv), "such that (iii) is satisfied". The authors have *already* postulated (iii), in the preceding item, so what would be the point of stating it again? After some thought, I decided that this was probably just another instance of the ideosyncratic exposition typical of all the DJ papers. Not until I saw DJ3 did I have any inkling of what the authors apparently meant, and it is so strange that I am amazed that they might expect that anyone could even guess it. (I never even considered the possibility, and had I considered it, I would have rejected it as too outlandish.)

Their (possible!) meaning became clear to me on p. 13 of DJ3. They observe that for the counterexample, the lowest nonzero order in g is $n = 1$. Then they *truncate* the power series $\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g)$ to first order to obtain from the original positive-operator valued measure (POVM) $\{\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g)\}_{j=1}^{3}$ three new operators $\hat{\boldsymbol{E}}_j^{'}(g)$ which are *linear* functions of g.[2] For example, $\hat{\boldsymbol{E}}_3^{'} = (1/6)I - gI$.

Then they assume that there should exist functions $\alpha_j(g)$ (not necessarily the original contextual values but confusingly denoted by the same symbol in DJ3)[3]

$$\sum_j \alpha_j(g) \hat{\boldsymbol{E}}_j^{'}(g) = \hat{\boldsymbol{A}} \quad ! \tag{$*$}$$

This is a strange and very restrictive assumption for which DJ3 gives no physical justification. (It also does not follow from (iv) as stated.)

To explain this, I need to emphasize a point discussed more fully in P1 (Section 7, p. 11), that contextual values (defined in general as functions $\alpha_j(g)$ satisfying the contextual value equation $\sum_j \alpha_j(g) \hat{\boldsymbol{E}}_{\boldsymbol{j}}(g) = \hat{\boldsymbol{A}}$) need not exist.

[2]Note that nothing in (iv) mentions truncation!

[3]The main difference between this version 2 and the original is that originally I didn't notice that the symbols α_j in the equation below might not represent the original contextual values. This discovery necessitated changing some subsequent wording, though not the conclusions.

This point is not explicitly mentioned in DAJ and not emphasized in DJ3. The truncated equation (*) is an example; this equation has no analytic solution $\{\alpha_j(g)\}$ for constant $\hat{\boldsymbol{A}} \neq 0$, and this is DJ3's objection to the counterexample. But on what physical grounds would one expect it to have a solution? Why is this a reasonable hypothesis for a "general theorem"?

It is clear that DJ3's "General theorem" is not very "general" after the original hypothesis (iv) is modified to include (*). It does not cover even simple cases to which one would expect to be able to apply a "general theorem", such as the measurement operators given in the second counterexample of P2 (reprinted in equation (7.1) of DJ3), for which the associated POVM is quadratic in g.

Added in Version 2:

After submitting the above, I thought of another conceivable interpretation of hypothesis (iv). Although in the end it seems untenable, I mention it in case some referee might be independently considering it. Could (iv) mean the following?

> There exists a least positive integer n such that (iii) holds for *some* functions $\alpha_j(g)$ (not necessarily the original contextual values as would be implied by a strict logical interpretation of the original (iv)) with the $\hat{\boldsymbol{E}}_j(g)$ replaced by their truncations $\hat{\boldsymbol{E}}_j^{'}(g)$ to order n.

This would a strange hypothesis which would be confusingly stated in the original (iv), but at least it makes sense, and would explain the puzzling "such that" in the original statement. However, under this interpretation, the objection to the counterexample would seem to make no sense, since this would define the integer n for the counterexample to be $n = 2$ rather than the $n = 1$ claimed by DJ3's analysis of the counterexample.

Also, the details of DJ3's attempted proof of its "General theorem" support the original interpretation of the phrase "minimum nonzero order". Whatever the case, the wording of (iv) should be sharpened to make the hypotheses for the "General theorem" clear and unambiguous.

5 The attempted proof of DJ3's "General theorem"

The last section noted that not only is there no apparent physical or mathematical reason to assume (*), but the exposition of DJ3 is so unclear that no reader could reasonably be expected to guess that the authors are making this assumption.

However, suppose we overlook these inconveniences and assume (iv) as including (*). Is the "General theorem" of Section 5 of DJ3 then valid?

I do not know. All I can say is that I cannot follow their attempted proof in detail, and I think that it contains a serious gap. I will be happy to discuss the gap in detail with any referee who is prepared to discuss the proof in detail. Since my questions/objections concerning it are quite technical, I do not want to take the time to write them down unless I am sure that someone will read them in detail.

Let us suppose that the gap can be filled so that DJ3's "General theorem" can be definitively proved. Here is another way to think about its conclusion that their "general conditioned average" is necessarily given by their (1.2) in the "minimal disturbance limit". For simplicity of exposition, I will consider only the case in which all measurement operators $\hat{\boldsymbol{M}}_{\boldsymbol{j}}$ are positive, and so are uniquely determined by the POVM $\{\hat{\boldsymbol{E}}_{\boldsymbol{j}}\}$ by $\hat{\boldsymbol{M}}_{\boldsymbol{j}} = \sqrt{\hat{\boldsymbol{E}}_{\boldsymbol{j}}}$. Also for simplicity, I restrict the discussion to the case in which the minimum nonzero order $n = 1$.

We are given a POVM $\{\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g\})$, which, via the contextual value equation and the pseudo-inverse prescription, uniquely determines a collection of contextual values $\{\alpha_j(g)\}$ (assuming that contextual values exist at all). The expansion of hypothesis (iv) of the theorem as including (*) gives a new POVM $\{\hat{\boldsymbol{E}}^{\prime}_j(g)\}$ whose elements $\hat{\boldsymbol{E}}^{\prime}_j(g)$ are *linear* in g, and which is assumed to also determine a set of (possibly different) contextual values.[4] If the gap in their proof can be filled, the proof will establish their (1.2).

So, assuming that DJ3's proof of its "General theorem" can be completed under the new hypothesis (*) for what DJ3 characterizes as the "typical case $n = 1$", it will essentially have as its main hypothesis that the POVM $\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g)$ be *linear* in g. More precisely, it would be valid only for nonlinear POVM's which could be replaced by a linear POVM. This would hold some interest, but would be nothing close to a justification of the original sweeping claim of DAJ that its "general conditioned average ... converges uniquely to the quantum weak value in the minimal disturbance limit", though DJ3 gives the misleading impression that it has justified this claim.

6 Summary and perspective

1. This paper does not refute the counterexamples of P2 to the sweeping claim of DAJ that

   > "[its 'general conditioned average'] converges uniquely to the quantum weak value in the minimal disturbance limit".

   The authors have added strong additional hypotheses to this claim to invalidate the counterexamples. Without these additional hypotheses, the authors do not dispute the counterexamples.

   Adding the hypothesis that the contextual values must be chosen to satisfy the "pseudo-inverse prescription" does invalidate the first counterexample. The second counterexample *does* satisfy the pseudo-inverse construction, but the authors claim incorrectly that it does not satisfy (iv) (*as stated* in

[4] Though the new POVM will produce the same average value as the original, it conceivably could be inferior in some way, such as having higher variance. It is not clear what would be the utility of the new POVM, nor why it is reasonable to assume that it should exist.

DJ1 and DJ3).

Hypothesis (iv) is so unclearly stated in DJ1, DJ2, and DJ3, that it is no exaggeration to say that no reader could possibly guess what the authors meant. What they did mean (presumably!) becomes clear only in DJ3's subsequent analysis of the counterexample, in which they reinterpret the actual language of (iv) to include condition (*) above. Under this modification, (iv) becomes so strong that it excludes even simple cases, such as the second counterexample whose POVM is quadratic.

2. The conclusion of DJ3's "General theorem" is not obvious even for the linear case. I do not believe that they have proved even the linear case, due to a major gap in the proof.

3. The paper repeats many questionable claims of DAJ and DJ1 which have previously been criticized in P1 and P2, without responding to that criticism.

4. The study of weak values was initiated in reference [1] of DJ3 by Aharonov, Albert, and Vaidman. They obtained (by questionable mathematics) the "weak value" stated in (1.1) of DJ3. Since this is not necessarily real but is supposed to represent a quantity (the shift of a pointer) which *is* manifestly real, most subsequent authors replace this by its real part, (which has also been claimed to have been derived by various means). I will refer to the real part of (1.1) as the "traditional" weak value; DJ calls it the "quantum weak value". For pure states, it is given by DJ3's equation (1.2).

   Recently other authors have obtained (at least one via rigorous mathematics) weak values different from the traditional weak value.[5] However, most of the "weak value" literature seems to implicitly assume that the only possible weak value is the traditional one, and to the best of my knowledge, only the traditional weak value has been observed in actual experiments. Why is this?

   A principal motivation of DAJ and all of the DJ papers seems to be to answer this question. DJ3 presents its "General theorem" as a satisfactory answer. I think that its hypotheses including (*) are far too strong to consider it a satisfactory answer. The original sweeping claim of DAJ that its "general conditioned average" "converges uniquely to the quantum weak value in the minimal disturbance limit" has been replaced by a claimed theorem with hypotheses so strong that even simple cases such as the second counterexample with a quadratic POVM are excluded.

[5]Cf. references [10] of DJ3, some of which are relevant.