Appendix

Calculation of equation (9.21) and further discussion

Note: A file similar to this appendix was sent to the authors. The discussion below was originally directed to them.

The aim is to calculate what I think is the correct version of equation (9.21), page 220 of Gerry/Knight. I'll use the text's notation.

The text considers a downconverter which outputs two photons called signal (s) and idler (i), both with the same polarization, "horizontal" (H). The idler photon is fed into a rotator which rotates its polarization to "vertical" (V).

The signal and idler photons are then are fed into the two ports of a beam splitter. The input state is denoted in equation (9.19) as $|H\rangle_s|V\rangle_i$. The precise meaning of this notation is not defined in the text, but I assume it means

$$|H\rangle_s|V\rangle_i = \hat{a}^\dagger_{s,H}\hat{a}^\dagger_{i,V}|0\rangle \quad , \tag{1}$$

where $\hat{a}^\dagger_{s,H}$ denotes the creation operator for a horizontally polarized photon in the signal beam, with similar notation for the idler beam. Also, $|0\rangle$ denotes the vacuum state for the input.

Following Section 6.2, the two output channels will be denoted as channels 1 and 2, with creation operators

$$\hat{a}^\dagger_{1,H} \ , \ \hat{a}^\dagger_{2,H} \ , \ \hat{a}^\dagger_{1,V} \ , \text{ and } \hat{a}^\dagger_{2,V} \quad .$$

I had to add a subscript specifying horizontal or vertical polarization, since the discussions of Chapter 6 didn't consider polarization. To avoid obscuring the simplicity of the matter with the notational complications of polarization, let's temporarily adopt Chapter 6's approach which suppresses the notation of polarization.

The formalism of Chapter 6 replaces the input creation operators with linear combinations of the output creation operators. The particular linear combination used is characteristic of the beam splitter. For the beam splitter considered here, equation (6.10) on page 139 (cf. also (6.17) on p. 141) gives these linear combinations as:

$$\begin{aligned} \hat{a}^\dagger_1 &= \frac{1}{\sqrt{2}}(\hat{a}^\dagger_s - i\hat{a}^\dagger_i) \\ \hat{a}^\dagger_2 &= \frac{1}{\sqrt{2}}(-i\hat{a}^\dagger_s + \hat{a}^\dagger_i) \\ &. \end{aligned} \tag{2}$$

Inverting this linear system to obtain the input creation operators in terms of the output creation operators gives:

$$\begin{aligned} \hat{a}^\dagger_s &= \frac{1}{\sqrt{2}}(\hat{a}^\dagger_1 + i\hat{a}^\dagger_2) \\ \hat{a}^\dagger_i &= \frac{1}{\sqrt{2}}(i\hat{a}^\dagger_1 + \hat{a}^\dagger_2) \quad . \end{aligned} \tag{3}$$

When polarization is included, the same linear combinations are used, with a polarization subscript H or V appended to both input and output creation operators.

The output state $|\psi_{out}\rangle$ is obtained by the replacement of $|H\rangle_s = \hat{a}^\dagger_{s,H}|0\rangle$ in (1) by

$$\frac{1}{\sqrt{2}}(\hat{a}^\dagger_{1,H} + i\hat{a}_{2,H})|0\rangle = \frac{1}{\sqrt{2}}(|H\rangle_1 + i|H\rangle_2) \quad ,$$

with a similar replacement for $|V\rangle_i$:

$$\begin{aligned} |\psi_{out}\rangle &= \frac{1}{\sqrt{2}}(|H\rangle_1 + i|H\rangle_2)\frac{1}{\sqrt{2}}(i|V\rangle_1 + |V\rangle_2) \\ &= \frac{1}{2}[i|H\rangle_1|V\rangle_1 + |H\rangle_1|V\rangle_2 - |H\rangle_2|V\rangle_1 + i|H\rangle_2|V\rangle_2] \quad . \end{aligned} \tag{4}$$

Note: The text usually writes a "product" state like $|H\rangle_2|V\rangle_1$ with the output channel 1 factor first, i.e., $|V\rangle_1|H\rangle_2$. That the two expressions should stand for the same thing is suggested by (1) and the fact that creation operators commute.

Also, sometimes a term like $|H\rangle_2|V\rangle_2$, denoting two photons in output channel 2 (one polarized horizontally, the other vertically) and none in output channel 1, is rendered in the text by notation like $|0\rangle_1|H\rangle_2|V\rangle_2$, (e.g., in equation (6.17) and the footnote on p. 220 explaining (9.21)).

Since the text never systematically explains its notational conventions, some guesswork is required in interpreting its expressions. Usually the physical meaning seems clear, as here, but I'm not sure the notation is mathematically fully consistent. I think a more systematic presentation of the formalism used would make the text easier to read, both for mathematically sophisticated readers and for those with more sketchy backgrounds.

The text's equation (9.21) instead gives the output state as

$$|\psi_{out}\rangle = \frac{1}{\sqrt{2}}[|H\rangle_1|V\rangle_2 - |V\rangle_1|H\rangle_2]$$

I think that $|\psi_{out}\rangle$ should be given by (4) rather than as just above. If so, the subsequent statement that "neither detector ... will fire alone is wrong. According to (4), detector 1 will fire alone (triggered by two photons, one horizontal and one vertical) 1/4 the time, and similarly for detector 2.

Following are some other comments.

1. I had quite a bit of trouble understanding what is the text's characterization of "interference" in contexts such as this. The term (which seems never to be precisely defined) brings to mind the interference fringes of a

two-slit experiment, but in the present context there are no such fringes—the experiment just counts simultaneous arrival of photons.

After a good deal of thought, I arrived at the following informal definition. It makes physical sense to me, but I don't know if it's what the text means. I'll explain it in the context of this experiment.

Suppose that the photons were classical noninteracting particles, like billiard balls which never collide. Then simple probability theory can be used to calculate the probability that both detectors register a particle.

This probability is $1/2$. Each particle has a $1/2$ probability of being transmitted (T) or reflected (R), and the transmission or reflection of one particle is assumed independent of that of the other particle (because the particles are noninteracting). Hence there are four possible outcomes, which we may abbreviate TT, TR, RT, RR, where the first letter describes what happens to the signal photon, and the second letter similarly describes the idler. By independence, each outcome has probability $1/4 = 1/2 \times 1/2$. The outcomes in which both detectors fire are TT and RR, so this event has probability $1/4 + 1/4 = 1/2$.

It seems reasonable to say that the photon experiment exhibits *no* "interference" if the photons act like classical billiard balls. In the present context, that implies that the probability of both detectors firing is $1/2$, and that is indeed what (4) implies. If the photon probabilities differ in any way from the billiard-ball probabilities, we can say that this demonstrates "interference".

I think that some explanation along these lines would be very helpful to the readers. The impression I got from the paragraph on p. 221 following (9.21) was that neither detector firing alone was somehow the hallmark of noninterference. But if we accept the above definition, then neither detector firing alone would actually *demonstrate* interference, because for the billiard balls there is a probability of $1/2$ that one detector fires alone (i.e., outcomes TR and RT).

2. The discussion of "decoherence" at the end of the paragraph following (9.21) is a mystery to me, as is virtually all of Chapter 8 on decoherence. Time evolution in quantum mechanics is unitary. If a system is in a pure state at some given time, then its state at a later time is obtained by applying a unitary operator to the original state.

   A unitary operator (in fact, any 1:1 operator) maps pure states to pure states by definition. Hence a system in a pure state *always* remains in a pure state. It cannot change into a "statistical mixture where only probabilities, not probability amplitudes" appear. This is an integral part of the mathematical structure of quantum mechanics.

   So why is it surprising that "in the case of the two-photon interferometry experiment discussed here, there is no time ... when we do not have a pure state"?