Comment on "Contextual Values of Observables in Quantum Measurements"

The Letter [1] of Dressel, Agarwal, and Jordan (henceforth called DAJ) introduces the concept of "contextual values" (CV) and claims that it leads to "a natural definition of a general conditioned average that converges uniquely to the quantum weak value in the minimal disturbance limit". They never precisely define their "minimal disturbance limit", but they do seem to define "quantum weak value" A_w in their equation (7):

$$A_w = \frac{\mathrm{Tr}\ [\hat{\mathbf{E}}_f^{(2)}\{\hat{\boldsymbol{A}}, \hat{\boldsymbol{\rho}}\}]}{2\mathrm{Tr}\ [\hat{\mathbf{E}}_f^{(2)}\hat{\boldsymbol{\rho}}]} \quad . \tag{7}$$

For pure states $\hat{\boldsymbol{\rho}}$, this reduces to the real part of the weak value obtained by the seminal paper [2].

A major error occurs in the passage from their definition (6) of "the conditioned average of an observable" to the quantum weak value (7). Introducing a small "measurement strength" parameter g, they write the measurement operators $\hat{\boldsymbol{M}}_{\boldsymbol{j}}$ as $\hat{\boldsymbol{M}}_{\boldsymbol{j}}(g) = \hat{\boldsymbol{U}}_{\boldsymbol{j}}(g)\hat{\boldsymbol{E}}_{\boldsymbol{j}}{}^{1/2}(g)$, with $\hat{\boldsymbol{U}}_{\boldsymbol{j}}$ unitary and $\hat{\boldsymbol{E}}_{\boldsymbol{j}}$ positive. This polar decomposition is surely possible, but they go on to attempt to apply Stone's theorem to write $\hat{\boldsymbol{U}}_{\boldsymbol{j}}(g) = \exp[ig\hat{\boldsymbol{G}}_{\boldsymbol{j}}]$. But Stone's theorem requires that $g \rightarrow \hat{\boldsymbol{U}}_{\boldsymbol{j}}(g)$ be a one-parameter unitary *group*, i.e., that $\hat{\boldsymbol{U}}_{\boldsymbol{j}}(g_1 + g_2) = \hat{\boldsymbol{U}}_{\boldsymbol{j}}(g_1)\hat{\boldsymbol{U}}_{\boldsymbol{j}}(g_2)$, which is surely not true in any generality in DAJ's context.

If $\hat{\boldsymbol{U}}_{\boldsymbol{j}}(g) = \exp[ig\hat{\boldsymbol{G}}_{\boldsymbol{j}}]$ were added as an *assumption*, it would be an extremely strong assumption which would be expected to hold for nontrivial $\hat{\boldsymbol{U}}_{\boldsymbol{j}}(g)$ only in unusual special cases. *A priori*, the unitary parts $\hat{\boldsymbol{U}}_{\boldsymbol{j}}$ of the measurement operators can be completely arbitrary. There is no reason to assume that for fixed j, the different $\hat{\boldsymbol{U}}_{\boldsymbol{j}}(g)$ should even commute, as they must if $\hat{\boldsymbol{U}}_{\boldsymbol{j}}(g) = \exp[ig\hat{\boldsymbol{G}}_{\boldsymbol{j}}]$.

The assumption $\hat{\boldsymbol{U}}_{\boldsymbol{j}}(g) = \exp(ig\hat{\boldsymbol{G}}_{\boldsymbol{j}})$ is so strong that the essence of the hypothesis for DAJ's claim to have established (7) is probably that the measurement operators be positive (i.e., that the $\hat{\boldsymbol{U}}_{\boldsymbol{j}}$ be trivial). This is what the book of Wiseman and Milburn [3] calls "minimally disturbing measurements", and the authors have confirmed[1] that they are using a slight generalization of this definition. To avoid complications, this Comment will assume Wiseman and Milburn's definition of "minimally disturbing measurement".

There is no limit in this definition, so DAJ's "minimal disturbance limit" still requires further definition. DAJ says that it derives (7) by taking "the weak limit of (6)", but does not define "weak limit". Subsequently,[2] the authors have defined "ideally weak measurement" as one satisfying

$$\lim_{g\rightarrow 0} \frac{\hat{\boldsymbol{M}}_{\boldsymbol{j}}(g)\, \hat{\boldsymbol{\rho}}\, \hat{\boldsymbol{M}}_{\boldsymbol{j}}{}^{\dagger}(g)}{\mathrm{Tr}\ [\hat{\boldsymbol{M}}_{\boldsymbol{j}}(g)\, \hat{\boldsymbol{\rho}}\, \hat{\boldsymbol{M}}_{\boldsymbol{j}}{}^{\dagger}(g)]} = \hat{\boldsymbol{\rho}} \quad \text{for all } j.$$

This says that in the limit of small g, the post-measurement state approaches the pre-measurement state $\hat{\boldsymbol{\rho}}$.

There is an example [4] of positive measurement operators $\hat{\boldsymbol{M}}_{\boldsymbol{j}}$ together with the other mathematical objects mentioned in (7) such that DAJ's "general conditioned average" [their equation (6)] is *not* the "quantum weak value" (7) in the limit of "ideally weak measurement".

Since an "ideally weak measurement" is presumably a special case of a "weak measurement" and the case of positive measurement operators is a special case of DAJ's slightly more general definition of "minimally disturbing measurement" this also seems a counterexample to DAJ's assertion that their "general conditioned average ... converges uniquely to the quantum weak value in the minimal disturbance limit".

S. Parrott
Dept. of Mathematics (retired), Univ. of Massachusetts at Boston, USA
Present address: 2575 Bowers Rd., Gardnerville, NV 89410, USA, S_Parrott@toast2.net

[1] J. Dressel, S. Agarwal, and A. N. Jordan, Phys. Rev. Lett. **104** 240401 (2010)

[1] Reply to Physical Review Letters to an earlier version of this Comment. Direct requests to the authors for their definitions have been ignored.

[2] ibid.

[2] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett **60**, 1351-1354 (1988)
[3] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control*, Cambridge University Press, 2009
[4] S. Parrott, www.arXiv.org/quant-ph/1102.4407v4