Comment on "Contextual Values of Observables in Quantum Measurements"

The Letter [1] of Dressel, Agarwal, and Jordan (henceforth called DAJ) introduces the concept of "contextual values" and claims that it leads to "a natural definition of a general conditioned average that converges uniquely to the quantum weak value in the minimal disturbance limit". This claim is false.

They never precisely define their "minimal disturbance limit", but they do seem to define"quantum weak value" A_w in their equation (7):

$$A_w = \frac{(1/2)\text{Tr }[\hat{\boldsymbol{P}}_f\{\hat{\boldsymbol{A}},\hat{\boldsymbol{\rho}}\}]}{\text{Tr }[\hat{\boldsymbol{P}}_f\hat{\boldsymbol{\rho}}]} \quad . \tag{7}$$

We use DAJ's notation and equation numbers throughout except that we replace DAJ's $\hat{\boldsymbol{E}}_f^{(2)}$, which represents the projector on a vector f, by the simpler $\hat{\boldsymbol{P}}_f$, $\hat{\boldsymbol{E}}_{jf}^{(1,2)}$ by $\hat{\boldsymbol{M}}_j^\dagger\hat{\boldsymbol{P}}_f\hat{\boldsymbol{M}}_j$ and their contextual values $\alpha_j^{(1)}$ by α_j.

DAJ's definition of "minimal disturbance limit" is not clearly given in the paper, but the authors have confirmed that their definition is implied by the simpler definition that their measurement operators $\hat{\boldsymbol{M}}_{\boldsymbol{j}}$ be positive and the measurement be "weak" in the sense that $\lim_{g\to 0}\ \hat{\boldsymbol{M}}_{\boldsymbol{j}}(g)\hat{\boldsymbol{\rho}}\hat{\boldsymbol{M}}_{\boldsymbol{j}}^\dagger(g)/\text{Tr }[\hat{\boldsymbol{M}}_{\boldsymbol{j}}(g)\hat{\boldsymbol{\rho}}\hat{\boldsymbol{M}}_{\boldsymbol{j}}^\dagger(g)] = \hat{\boldsymbol{\rho}}$, which will be assumed here.

For positive (hence Hermitian) measurement operators $\hat{\boldsymbol{M}}_j$, DAJ's "general conditioned average" ${}_f\langle\mathcal{A}\rangle$ is defined by:

$${}_f\langle\mathcal{A}\rangle = \frac{\sum_j \alpha_j \text{Tr }[\hat{\boldsymbol{M}}_{\boldsymbol{j}}\hat{\boldsymbol{P}}_f\hat{\boldsymbol{M}}_{\boldsymbol{j}}\hat{\boldsymbol{\rho}}]}{\sum_k \text{Tr }[\hat{\boldsymbol{M}}_k\hat{\boldsymbol{P}}_f\hat{\boldsymbol{M}}_k\hat{\boldsymbol{\rho}}]}. \tag{6}$$

Using the identity $\hat{\boldsymbol{M}}_{\boldsymbol{j}}\hat{\boldsymbol{\rho}}\hat{\boldsymbol{M}}_{\boldsymbol{j}} = (1/2)\{\hat{\boldsymbol{M}}_j^2,\hat{\boldsymbol{\rho}}\} + 1/2[\hat{\boldsymbol{M}}_{\boldsymbol{j}},[\hat{\boldsymbol{\rho}},\hat{\boldsymbol{M}}_{\boldsymbol{j}}]]$, the numerator of (6) decomposes into a corresponding sum of two terms, the first of which yields (7) in the weak limit. Since we agree with DAJ on the weak limit of the denominator, namely Tr $[\hat{\boldsymbol{P}}_f\hat{\boldsymbol{\rho}}]$, and the first term of the numerator, for brevity we omit these calculations. The counterexample consists of showing that the weak limit of the second term,

$$\frac{\sum_j \alpha_j^{(1)}\text{Tr }[[\hat{\boldsymbol{M}}_{\boldsymbol{j}},[\hat{\boldsymbol{\rho}},\hat{\boldsymbol{M}}_{\boldsymbol{j}}]\hat{\boldsymbol{P}}_f]}{\sum_k \text{Tr }[\hat{\boldsymbol{M}}_k\hat{\boldsymbol{P}}_f\hat{\boldsymbol{M}}_k\hat{\boldsymbol{\rho}}]} \tag{$*$}$$

need not vanish.

The counterexample uses three diagonal measurement operators which are 2×2 matrices, $\hat{\boldsymbol{M}}_1(g) := \text{diag}\{1/2+g, 1/2-g\}, \hat{\boldsymbol{M}}_2(g) := \hat{\boldsymbol{M}}_1(-g)$, and $\hat{\boldsymbol{M}}_3(g) := \sqrt{1/2-2g^2}\hat{\boldsymbol{I}}$, where $\hat{\boldsymbol{I}}$ denotes the identity matrix. The "system observable" $\hat{\boldsymbol{A}}$ will also be a diagonal matrix $\hat{\boldsymbol{A}} := \text{diag}\{a,b\}$. The calculations are particularly simple for $\hat{\boldsymbol{A}} := \hat{\boldsymbol{I}}$, which is not physically interesting but still a mathematically valid counterexample, so that is what we shall assume. Messier calculations produce counterexamples for any diagonal $\hat{\boldsymbol{A}}$.

The "contextual values" $\alpha_j = \alpha_j(g)$ satisfy DAJ's equation (4): $\sum_j \alpha_j \hat{\boldsymbol{M}}_{\boldsymbol{j}}^2 = \hat{\boldsymbol{A}}$. Writing this out in components gives a system of two equations in three unknowns α_j which happen to coincide when $\hat{\boldsymbol{A}} = \hat{\boldsymbol{I}}$ and $\alpha_2 = \alpha_1$:

$$(1/2+2g^2)\alpha_1(g) + (1/2-2g^2)\alpha_3(g) = 1.$$

The counterexample uses a solution with $\alpha_2(g) = \alpha_1(g) := 1/g^2$. Then one routinely calculates that the weak limit of ($*$) for $\hat{\boldsymbol{\rho}} = (\rho_{ij})$ is

$$\frac{-\text{Tr }[\hat{\boldsymbol{P}}_f \begin{bmatrix} 0 & 4\rho_{12} \\ 4\rho_{21} & 0 \end{bmatrix}]}{\text{Tr }[\hat{\boldsymbol{P}}_f\hat{\boldsymbol{\rho}}]},$$

which does not always vanish. To compress this Comment to the one-page limit, routine calculations had to be omitted, but they can be found in full along with more user-friendly discussion in [2] and [3].

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[1] J. Dressel, S. Agarwal, and A. N. Jordan, Phys. Rev. Lett. **104** 240401 (2010)
[2] S. Parrott, arXiv:1105.4188
[3] S. Parrott, arXiv:1102.4407v6