Asymptotics of a proposed delay-differential equation of motion for charged particles

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Abstract

F. Rohrlich has recently published two papers advocating a particular delay-differential (DD) equation as an approximate equation of motion for classical charged particles, which he characterizes as providing a "fully acceptable classical electrodynamics". We study the behavior in the remote past and future of solutions of this equation for the special case in which the motion is in one spatial dimension.

We show that if an external force is applied for a finite time, some solutions exhibit the property of "preacceleration", meaning that the particle accelerates *before* the force is applied, but that there do exist solutions without preacceleration. However, most solutions without preacceleration exhibit "postacceleration" into the infinite future, meaning that the particle accelerates *after* the force is removed. Some may regard such behavior as sufficiently "unphysical" to rule out the equation.

More encouragingly, analogs of the pathological "runaway" solutions of the Lorentz-Dirac equation do not occur for solutions of the DD equation. We show that when the external force eventually vanishes, the proper acceleration vanishes asymptotically in the future, and the coordinate velocity becomes asymptotically constant.

1 Introduction

The correct equation to describe the motion of a charged particle in flat spacetime (Minkowski space) has long been a matter of speculation and controversy. The most-mentioned candidate has been the Lorentz-Dirac equation, written here for units in which light has unit velocity and metric tensor of signature [+, -, -, -]:

$$m\frac{du^{i}}{d\tau} = qF^{i}{}_{\alpha}u^{\alpha} + \frac{2}{3}q^{2}\left[\frac{d^{2}u^{i}}{d\tau^{2}} + \frac{du^{\alpha}}{d\tau}\frac{du_{\alpha}}{d\tau}u^{i}\right]$$
(1)

This is written in traditional tensor notation with repeated indices summed and emphasized in Greek; $u = u^i$ denotes the particle's four-velocity, m and q its mass and charge, respectively, τ its proper time, and F_{ij} the (antisymmetric) tensor describing an external electromagnetic field driving the motion.

However, many objections have been raised to this equation. Among them are the existence of "runaway" solutions in which the acceleration increases exponentially with proper time even when the external field asymptotically vanishes. In some physically reasonable situations, *all* solutions are runaway (Eliezer, 1943; for later references see Parrott, 1987, Section 5.5). Even in favorable cases in which non-runaway solutions exist, they may exhibit so-called "preacceleration" in which the particle begins to accelerate *before* the external field is applied (Rohrlich, 1965).

F. Rohrlich (1997, 1999) has recently advocated a new equation of motion which (Rohrlich, 1999) claims without proof "[has] no pathological solutions". This new equation is a delay-differential equation given below, which we shall call the "DD equation". The present paper derives from a study of the DD equation with the aim of determining the correctness of such claims.

(Rohrlich, 1997) claims that the DD equation has neither preaccelerative nor "runaway" solutions. We shall show that for the case of a nonzero external force applied for only a finite time, the DD equation does admit preaccelerative solutions, and that these are all "runaway" in the weak sense that the acceleration does not vanish asymptotically in the distant past as one would expect. However, this does not rule out the DD equation as a realistic equation of motion because we also show that these preaccelerative solutions can be eliminated by appropriate choice of generalized initial conditions (defined below). But then another problem arises: assuming this choice, most solutions exhibit "postacceleration", meaning that the acceleration persists after the external field is turned off.

Postacceleration is not as bad as preacceleration because there is no violation of causality, but postaccelerative solutions could be considered pathological. Suppose we are sitting in a room shielded from electromagnetic fields watching a beam of identical charged particles shoot in the window. It might seem strange if some of the particles speeded up, while others slowed down, for no apparent reason, according to their past histories. This is what the DD equation predicts.

(Rohrlich, 1997) presents the DD equation as a modification of the following equation proposed by (Caldirola, 1956):

$$\frac{m}{\tau_0}[u^i(\tau - \tau_0) - u^\alpha(\tau - \tau_0)u_\alpha(\tau)u^i(\tau)] + \frac{q}{c}F^i{}_\alpha u^\alpha(\tau) = 0 \quad .$$
(2)

This is written in units in which the velocity of light is c, and $\tau_0 := 4q^2/(3mc^3) \approx 1.2 \times 10^{-23}$ sec is a constant with the dimensions of time. With appropriate units

employing our convention that the velocity of light is unity, this can be written:

$$\frac{m}{\tau_0} [u^i(\tau - \tau_0) - u^\alpha(\tau - \tau_0)u_\alpha(\tau)u^i(\tau)] + qF^i{}_\alpha u^\alpha = 0 \quad . \tag{3}$$

The motivation presented by (Caldirola, 1956) involves starting with the Lorentz equation (the classical force equation ignoring radiation reaction),

$$m\frac{du^i}{d\tau} = qF^i{}_{\alpha}u^{\alpha} \quad , \tag{4}$$

and attempting to replace $du^i/d\tau$ by the difference quotient

$$\frac{u^{i}(\tau) - u^{i}(\tau - \tau_{0})}{\tau_{0}} \quad . \tag{5}$$

But since both sides of (4) are orthogonal to u, while (5) need not be, one ought to project (5) into the subspace orthogonal to $u(\tau)$. This projection will kill any multiple of $u(\tau)$, so the resulting equation can be rewritten as (3)

Rohrlich (1997) suggested the following modification of (3). This is what we are calling the DD equation:

$$m_1 \frac{du^i}{d\tau} = f^i(\tau) + m_2 [u^i(\tau - \tau_1) - u^\alpha(\tau - \tau_1)u_\alpha(\tau)u^i(\tau)] \quad . \tag{6}$$

Here $f(\tau) = f^i(\tau)$ is a four-force orthogonal to $u(\tau)$, m_1 and m_2 are presumably nonzero parameters associated with his motivation of the right side as an approximation to the self-force on a spherical surface charge, and τ_1 is a positive parameter. (The sign of the second term in brackets differs from his because his metric is opposite in sign to ours.) To avoid degenerate cases, we will assume below that m_2 and m_1 are nonzero; actually, they would be expected to be positive.

(Rohrlich, 1997) attributes the conjecture of (6) to (Caldirola, 1956), but (6) is not mathematically equivalent to (3), and (Caldirola, 1956) did not propose it. For example, Caldirola's equation for a compactly supported force admits periodic solutions (to which Caldirola attached great importance as possibly describing spin-like internal particle motions), while the DD equation does not (as the analysis of this paper will make clear). Note also that the DD equation (6) is a delay-differential equation, while Caldirola's equation is a difference equation involving no derivatives.

The abstract of (Rohrlich, 1999), begins as follows:

"The self-force for the classical dynamics of finite size particles is obtained. It is to replace the one of von Laue type obtained for point particles. \dots "¹

(Rohrlich, 1999) emphasizes the contribution of (Yaghjian, 1992, Appendix D., Equation D.19). Yaghjian derived a rest-frame version of the DD equation as

¹The equation "of von Laue type" is the Lorentz Dirac equation (1).

an uncontrolled approximation 2 to the self-force on a charged particle modeled as a "rigid" sphere. 3

Yaghjian's equation is written for a particle slowly moving relative to some given Lorentz frame. If the given Lorentz frame is the particle's rest frame at a given proper time, and it is written in four-dimensional notation, then it becomes the DD equation. Thus the DD equation (6) is a relativistic generalization of Yaghjian's equation, and its statement in (Rohrlich, 1997) appears to be the first explicit statement in the literature. Moreover, (Rohrlich, 1997), goes beyond its mere statement to present it as a "fully acceptable classical electrodynamics". By comparison, Yaghjian obtained it as an approximation without comment on the approximation's domain of validity.

Because of this and the above quote from the abstract of (Rohrlich, 1999) stating that it "obtained" the equation, an earlier version of the present paper called the DD equation (6) "Rohrlich's equation". However, when Professor Rohrlich submitted a report as an identified referee for that version, he strongly objected to the name "Rohrlich's equation", attributing it instead to Caldirola and Yaghjian. In deference to his wishes, we sidestep the convoluted question of the origin of equation (6) by employing the neutral name "DD" equation.

Although Rohrlich's motivation for the DD equation (6) involves thinking of the particle as a charged sphere, the equation of motion itself is the equation for a moving point, the center of the sphere. Thus the DD equation is mathematically the equation of a point particle, and we shall consider it as such; the motivation of the charged sphere will not enter into our considerations.

To study (6), it will be convenient to eliminate inessential constants by introducing a new time unit equal to τ_1 old time units and by using available notational freedom to absorb the two parameters m_1 and m_2 into a single parameter m, obtaining

$$m\frac{du^{i}}{d\tau} = f^{i} + [u^{i}(\tau - 1) - u^{\alpha}(\tau - 1)u_{\alpha}(\tau)u^{i}(\tau)] \quad .$$
(7)

This can be done by dividing both sides of (6) by m_2 , and renaming f and m_1 .

The resulting equation (7), which we shall call the "normalized DD equation", is equivalent to the original DD equation (6) in the sense that knowledge of all solutions of either translates immediately into knowledge of all solutions of the other. In response to a referee's skepticism about this, we added Appendix 2, which works out in detail the precise relations between the two equations. These will also be briefly sketched just below for the reader's convenience.

The analysis of the paper *could* be written without introducing the normalized DD equation (7), and the conclusions would be essentially the same, apart

 $^{^2 {\}rm For}$ example, the derivation ignores nonlinear terms. "Uncontrolled" describes the fact that no attempt is made to quantitatively assess the magnitude of possible errors which such simplifications might introduce.)

 $^{^{3}}$ Yahgjian mentions that essentially the same equation was "stated without proof" by (Page, 1918), and gives references to other precursors. Also, Caldirola refers to precursors of his proposed equation. The history of these ideas is extremely tangled, and we make no attempt to present all aspects of it here.

from trivial changes of language. We chose not to do that because retaining the original, more complicated notation would only distract attention from the more fundamental ideas which we want to emphasize.

To avoid any misunderstanding, we make explicit that the unknown function "u" in (7) is technically not the same as the "u" in (6), but the two "u"s are related in a mathematically straightforward and physically transparent way given below in equation (8), so that conclusions about solutions of (7) can be directly translated into conclusions about solutions of (6), and vice versa.

More explicitly, changing to the new time unit corresponds to multiplying the old metric by τ_1^{-2} . This results in a new proper time $\bar{\tau} := \tau_1 \tau$ and a new four-velocity \bar{u} satisfying

$$\bar{u}(\bar{\tau}) = \tau_1 u(\tau) \quad . \tag{8}$$

Equivalently, for any real number s,

$$\bar{u}(s) = \tau_1 u(\tau_1 s) \quad . \tag{9}$$

This simple relation allows us to translate any conclusion about \bar{u} into a corresponding conclusion about u, and vice versa.

The original DD equation (6) is thus equivalent to an equation like (7) but with symbols renamed:

$$m\frac{d\bar{u}^{i}}{d\bar{\tau}} = \bar{f}^{i} + [\bar{u}^{i}(\bar{\tau}-1) - \bar{u}^{\alpha}(\bar{\tau}-1)\bar{u}_{\alpha}(\bar{\tau})\bar{u}^{i}(\bar{\tau})] \quad .$$
(10)

The details of the relations of the renamed symbols to the originals are given in Appendix 2.

In the rest of the paper, we omit the bars for typographical simplicity, and analyze (7) instead of (10), leaving to the reader the straightforward translation of mathematical results about (7) to conclusions about (6). All of our conclusions about (6) will be essentially identical to the corresponding conclusions about (10), apart from trivial changes in language or notation. For example, Theorem 5's conclusion that for eventually vanishing force, $\lim_{\bar{\tau}\to 0} d\bar{u}/d\bar{\tau} = 0$, translates into the conclusion that solutions u of (6) satisfy $\lim_{\tau\to 0} du/d\tau = 0$.

2 The DD equation for one-dimensional motion

We shall study the (normalized) DD equation (7) for the case of motion in one space dimension. (Henceforth we omit the reminder "normalized".) Analysis of this special case turns out to be sufficient to answer the questions motivating this work.

For this case, we may work in a two-dimensional Minkowski space with typical vector $x = (x^0, x^1)$ and metric tensor $g(x, x) = (x^0)^2 - (x^1)^2$. Since $u^{\alpha}u_{\alpha} = g(u, u) = 1$, we may write

$$u = (\cosh \theta, \sinh \theta) \quad , \tag{11}$$

where this defines the "rapidity" parameter θ . Define

$$w := (\sinh\theta, \cosh\theta) \quad , \tag{12}$$

so that w is a spacelike unit vector orthogonal to u, and any vector orthogonal to u must be a multiple of w.

In the DD equation (7), all of $du/d\tau$, f, and the bracketed term are orthogonal to u, and hence multiples of w. This can be seen explicitly for the left side:

$$m\frac{du}{d\tau} = m\frac{d\theta}{d\tau}w \quad .$$
$$A := \frac{d\theta}{d\tau} \tag{13}$$

It is natural to name

the "scalar proper acceleration".

Write

 $f = Ew \quad ,$

where this defines the scalar E. When $f^i = F^i{}_{\alpha}u^{\alpha}$ is a Lorentz force, E is the electric field (nominally relative to the Lorentz frame corresponding to the basis u, w for two-dimensional Minkowski space, but actually relative to the Lorentz frame determined by *any* orthogonal basis, as is revealed by straightforward calculation).

The bracketed term of (7), being a multiple of w, is equal to its projection on w. The projection of an arbitrary vector v on w is $-v^{\alpha}w_{\alpha}w$, so the projection of the bracketed term of (7) on w is w multiplied by the scalar

$$-(\cosh\theta(\tau-1)\sinh\theta(\tau)-\sinh\theta(\tau-1)\cosh\theta(\tau)=\sinh(\theta(\tau-1)-\theta(\tau))$$

Projecting the entire DD equation (7) on w yields the equivalent scalar equation

$$mA = m\frac{d\theta}{d\tau} = E + \sinh(\theta(\tau - 1) - \theta(\tau)) \quad . \tag{14}$$

We shall call this the variant of the DD equation for one-dimensional motion the "DD1 equation".

Equation (14) may be regarded as a delay-differential equations of the general form

$$\frac{d\lambda}{d\tau} = \Phi(\tau, \lambda(\tau), \lambda(\tau-1)) \quad , \tag{15}$$

with Φ a given function. If we imagine $E(\tau)$ as a given function of proper time τ , then (14) is of the form (15) with $\Phi(\tau, r, s) := E(\tau) + \sinh(r - s)$.)

In general, the situation is more complicated because θ is defined by (11), with $u^i := dz^i/d\tau$, where $z^i(\tau)$ represents the particle's wordline. Then E is usually not explicitly given as a function of proper time, but instead is given as a function $E(z, u) = E(z(\tau), u(\tau))$ of the Minkowski coordinates and four-velocity, with its proper time dependence acquired indirectly from the time-dependence of the latter. However, we may still regard the equation (14) as of the form (15) by imagining solving (14) for $\theta(\tau)$, which determines $u(\tau)$, then (by integration) $z(\tau)$, and finally $E(z(\tau), u(\tau))$. (The existence of the solution $u(\tau)$ will be established in the next section.) Following common abuse of notation, we write $E(\tau)$ in place of $E(z(\tau), u(\tau))$.

This shows that θ does satisfy *some* delay-differential equation of the form (15). We shall show that this severely restricts the form of the function θ . For example, we'll show that if $\tau \mapsto E(\tau)$ has compact support, then θ must be bounded on any semi-infinite interval $[\tau_0, \infty)$.

Similar remarks apply to the equation obtained by differentiating (14):

$$m\frac{dA}{d\tau} = \frac{dE}{d\tau} + \cosh(\theta(\tau - 1) - \theta(\tau))[A(\tau - 1) - A(\tau)] \quad . \tag{16}$$

This may also be regarded as of the form (15) if we imagine that we have already solved for θ . At first sight this may seem strange because if we have solved for θ , then we also have $A := d\theta/d\tau$, so there is no need to solve (16) for A. Nevertheless, the observation that A satisfies an equation of the form (16) is useful because it severely constrains A. For example, we'll show that it implies that for a force $E(\cdot)$ with compact support, $\lim_{\tau\to\infty} A(\tau)$ exists. Then combining this with the above-mentioned fact that θ is bounded will imply that in fact $\lim_{\tau\to\infty} A(\tau) = 0$.

3 General delay-differential equations of form (15)

This section reviews some simple and well-known facts about general delaydifferential equation of the form (15). When discussing such equations, we will always call them "delay-differential" equations, which will never be abbreviated. We reserve the term "DD equation" for the particular special case (7) or one of its variants such as (6).

Let the function Φ in the delay-differential equation (15) be C^1 (i.e., continuously differentiable). Suppose that $\tau \mapsto \lambda(\tau)$ satisfies this equation. If we regard $\lambda(\tau)$ as given on some interval [n-1, n], then (15) becomes an ordinary differential equation for λ on [n, n+1] of the form

$$\frac{d\lambda}{d\tau} = \Psi(\tau, \lambda(\tau)), \quad n \le \tau \le n+1 \quad , \tag{17}$$

which is covered by the standard existence and unigueness theorems.

This observation reveals the general structure of solutions of (15). Choose $\lambda(\cdot)$ to be an arbitrarily chosen C^1 function on any closed interval of length 1, say the interval [0, 1], subject to the consistency condition

$$\frac{d\lambda}{d\tau}(1) = \Phi(1,\lambda(1),\lambda(0)),\tag{18}$$

where the derivative in (18) is understood as a derivative from the left. Such a specification of λ on an interval of length 1 will be called a *generalized initial condition*.

Then (15) determines a unique solution $\lambda(\tau)$ on some maximal interval $1 \leq \tau < \delta$ with $\delta \leq 2$. We shall show below that for the equations of interest to us, namely (14) and (16), we actually have $\delta = 2$, and $\lambda(\cdot)$ satisfies the equation on [1, 2], where $\lambda'(2)$ is understood as a derivative from the left. Iterating the process yields a solution λ on $[0, \infty)$ whose values on [0, 1] can be an arbitrarily specified C^1 function satisfying the consistency condition (18).

Iterating to the left to obtain a unique solution on $(-\infty, 0]$ determined by λ restricted to [0, 1] involves inverting $s \mapsto \Phi(\tau + 1, \lambda(\tau + 1), s)$ for fixed τ . For equations (14) or (16), this is trivial. For example for (14), given $\theta(\tau)$ defined for $0 \leq \tau < 1$, simply define θ on the "preceding" interval $-1 \leq \tau < 0$

$$\theta(\tau) := \sinh^{-1}(m\frac{d\theta}{d\tau}(\tau+1) - E(\tau+1)) + \theta(\tau+1), \quad -1 \le \tau < 0,$$
(19)

where the derivative $d\theta/d\tau(\tau+1)$ is understood as a derivative from the right when $\tau + 1 = 0$.

4 Special cases of the general delay-differential equation (15)

4.1 Mathematical preliminaries

The delay-differential equation (15) relates the solution $\lambda(\cdot)$ on an interval $[\alpha, \alpha+1]$ to the solution on the "preceding" interval $[\alpha-1, \alpha]$. The following proposition shows that for a class of equations which includes (14) and (16) (the latter with θ regarded as given, *a priori*), the maximum of λ on an interval $[\alpha, \alpha+1]$ is no greater than the maximum on the preceding interval $[\alpha - 1, \alpha]$. Similarly the minimum of λ on $[\alpha, \alpha + 1]$ is no less than the minimum on the preceding interval.

For an arbitrary C^1 real-valued function λ on the real line, and arbitrary $\alpha < \beta,$ define:

$$M_{[\alpha,\beta]}(\lambda) := \max\{\lambda(\tau) \mid \alpha \le \tau \le \beta\}$$
(20)

$$m_{[\alpha,\beta]}(\lambda) := \min\{\lambda(\tau) \mid \alpha \le \tau \le \beta\}$$
(21)

Proposition 1 Let $\tau, s \mapsto \Omega(\tau, s), -\infty < \tau, s, < \infty$, be a C^1 function such that for each $\tau, s \mapsto \Omega(\tau, s)$ is a strictly increasing function satisfying $\Omega(\tau, 0) = 0$. Let λ be a solution of a delay-differential equation of the special form

$$\frac{d\lambda}{d\tau} = \Omega(\tau, \lambda(\tau - 1) - \lambda(\tau)) \quad . \tag{22}$$

Then for all α ,

$$M_{[\alpha,\alpha+1]} \leq M_{[\alpha-1,\alpha]} \tag{23}$$

$$m_{[\alpha,\alpha+1]} \geq m_{[\alpha-1,\alpha]} \quad . \tag{24}$$

Proof: For notational simplicity we take $\alpha := 0$. Thus we will prove that

$$M_{[1,2]} \le M_{[0,1]}$$

The proof of the corresponding assertion for m, which only requires reversing a few inequalities, will be omitted.

Let τ_{max} denote a point in [1, 2] with

$$\lambda(\tau_{\max}) = M_{[1,2]}(\lambda) \quad .$$

If τ_{max} is an interior point of [1, 2], then

$$0 = \lambda'(\tau_{\max}) = \Omega(\tau_{\max}, \lambda(\tau_{\max} - 1) - \lambda(\tau_{\max}))$$

and since $s \mapsto \Omega(\tau, s)$ is strictly increasing and zero only at s = 0, we have

$$M_{[0,1]} \ge \lambda(\tau_{\max} - 1) = \lambda(\tau_{\max}) = M_{[1,2]}$$

If $\tau_{\rm max} = 1$, then

$$M_{[0,1]} \ge \lambda(1) = \lambda(\tau_{\max}) = M_{[1,2]}$$

If $\tau_{\text{max}} = 2$, then we must have

$$\lambda'(2) \ge 0 \quad , \tag{25}$$

otherwise there would be points $\tau < 2$ with $\lambda(\tau) > \lambda(2)$ (i.e., the graph of λ would be going down at 2), contradicting $\lambda(2) = \lambda(\tau_{\max}) \geq \lambda(\tau)$ for all $1 \leq \tau \leq 2$. Now from (25) and (22),

$$0 \le \lambda'(2) = \Omega(2, \lambda(1) - \lambda(2))$$

so again using the fact that $\Omega(2, s)$ is increasing with $\Omega(2, 0) = 0$, we conclude that $\lambda(1) - \lambda(2) \ge 0$. Finally,

$$M_{[0,1]}(\lambda) \ge \lambda(1) \ge \lambda(2) = \lambda(\tau_{\max}) = M_{[1,2]}(\lambda).$$

Now we prove that given an arbitrary C^1 specification of $\lambda(\cdot)$ on the interval [0,1] satisfying the consistency condition (18), there exists a solution λ to (22) defined on $[0,\infty)$ and taking the specified values on [0,1]. By a C^1 function on [0,1], we mean a continuously differentiable function, with the derivatives at 0 and 1 understood as derivatives from the right and left, respectively.

Proposition 2 Let $\tau \mapsto \psi(\tau)$ be a C^1 function on [0,1] satisfying the consistency condition (18) Consider the equation (22) of Proposition 1:

$$\frac{d\lambda}{d\tau} = \Omega(\tau, \lambda(\tau - 1) - \lambda(\tau)) \quad , \tag{26}$$

where Ω satisfies the hypotheses of that Proposition. Then there exists a unique C^1 solution λ defined on $[0, \infty)$ and satisfying $\lambda(\tau) = \psi(\tau)$ for $0 \le \tau \le 1$.

Proof: If we regard $\lambda(\tau - 1) = \psi(\tau - 1)$ as given on $1 \leq \tau \leq 2$, then equation (26) is covered by the standard existence and uniqueness theorems for ordinary differential equations (e.g., Perko, 1996, Chapter 2). By these results, given the initial condition $\lambda(1) = \psi(1)$, there exists a maximal interval $[1, \delta)$, $1 < \delta \leq 2$, such that there exists a C^1 solution λ on $[1, \delta)$ satisfying that initial condition. Moreover, if $\lambda(\tau)$ remains in a compact set for $1 \leq \tau < \delta$, then $\delta = 2$ (Perko, 1996, Section 2.4, Theorem 2).

In other words, the only way the solution can fail to be globally defined is if it blows up. By Proposition 1, our solution does not blow up; hence it is defined for $0 \le \tau \le 2$. Iteration produces a solution defined on $[0, \infty)$.

We can also iterate to the left to obtain a solution defined on $(-\infty, \infty)$, assuming that for fixed τ we can invert $s \mapsto \Omega(\tau, s)$. The inversion is trivial for equations of the form (14) and (16), so we have:

Proposition 3 Let ψ be a C^1 function on [0,1] satisfying the consistency condition (18). Then for any C^1 function $\tau \mapsto E(\tau)$, there exists a C^1 solution $\tau \mapsto \theta(\tau)$ to the DD1 equation (14), defined for $-\infty < \tau < \infty$ and satisfying $\theta(\tau) = \psi(\tau)$ for $0 \le \tau \le 1$.

The same holds with A in place of θ for the acceleration equation (16), provided we regard θ as given a priori and assume that $E(\cdot)$ is C^2 , so that $dE/d\tau$ is C^1 .

4.2 Preacceleration and solutions runaway in the past

The simple propositions of the previous subsection imply quite a lot. Consider the DD1 equation (14)

$$mA = m\frac{d\theta}{d\tau} = E + \sinh(\theta(\tau - 1) - \theta(\tau))$$

for a continuous force E applied for only a finite time, say for $0 \le \tau \le \tau_f$ (i.e., $E(\tau)$ vanishes off this interval).

Suppose that A does not vanish identically on [-1, 0]. Then A cannot vanish asymptotically in the past because by Proposition 1,

$$M_{[-1,0]}(A) \le M_{[-2,-1]}(A) \le M_{[-3,-2]}(A) \le \dots$$

and

$$m_{[-1,0]}(A) \ge m_{[-2,-1]}(A) \ge m_{[-3,-2]}(A) \ge \dots$$

The first equation tells us that if $M_{[-1,0]}(A) > 0$, then A assumes values at least as large as this infinitely often in the distant past. If, on the other hand, $M_{[-1,0]}(A) \leq 0$, then also $m_{[-1,0]}(A) \leq M_{[-1,0]}(A) \leq 0$, so |A| assumes values at least at least as large as $|m_{[-1,0]}(A)|$ infinitely often in the distant past. So the only case in which A could vanish asymptotically in the distant past is if $m_{[-1,0]}(A) = M_{[-1,0]}(A) = 0$, but then A vanishes identically on [-1,0].

Thus the only physically reasonable initial specification of A on [-1, 0] is

$$A(\tau) \equiv 0 \quad \text{for } -1 \le \tau \le 0 \quad . \tag{27}$$

A glance at (16) shows that this implies that $A(\tau)$ vanishes identically for $\tau \leq 0$, and so θ is constant (i.e., the velocity is constant) for $\tau \leq 0$.

So, we see that even for a compactly supported force (indeed, even for identically zero force), there *are* solutions of the DD equation which do not vanish asymptotically in the past. These solutions also exhibit "preacceleration". However, for a compactly supported force, we can choose the initial specification (27) of A so as to eliminate these pathological solutions.

The situation for a force which is not compactly supported seems unclear. It does not seem obvious how to choose the initial specification so as to force A to vanish asymptotically as $\tau \to -\infty$ when E does. Since the theory is physically incomplete unless one gives a prescription for an initial specification of θ on some interval of length 1, this is a point which advocates of the DD equation should address.

Suppose we have agreed on such a prescription. We might define the A determined by the prescription as *preaccelerative* if there exist two force functions E_1 and E_2 with $E_1(\tau) = E_2(\tau)$ for $\tau \leq 0$, but $A_1(\tau) \neq A_2(\tau)$ for some $\tau < 0$, where A_1 and A_2 denote the corresponding accelerations obtained by solving the DD equation. It does not seem obvious that there is a prescription which will outlaw preaccelerative solutions.

4.3 Postacceleration

This section considers a force applied for only a finite time τ_f , say $0 \le \tau \le \tau_f$. For such a force, a *preaccelerative* solution of the DD equation (7) or DD1 equation (14) will be defined as one such that the acceleration A of the corresponding solution does not vanish identically on $(-\infty, 0]$, and a *postaccelerative* solution is one whose acceleration does not vanish identically on $[\tau_f, \infty)$.

The discussion to follow explains why either preacceleration or "postacceleration" is essentially built into the DD1 equation: one or the other must occur except in unusual special cases. This can be guessed from the observation that we have two ways to solve (uniquely) the DD1 equation (7):

$$mA = m\frac{d\theta}{d\tau} = E + \sinh(\theta(\tau - 1) - \theta(\tau)) \quad .$$
⁽²⁸⁾

We may solve (28) forward in time from a generalized initial condition $u(\tau) \equiv 0$ for $-1 \leq \tau \leq 0$. We can also solve backward in time from a "generalized final condition" $u(\tau) \equiv 0$ for $\tau_f \leq \tau \leq \tau_f + 1$. There is no reason to imagine that these two solutions will coincide, and we shall now show that they almost never do.

We may take $\tau_f = n$, with n a positive integer. Consider the DD1 equation (28) for $\tau_f = n \leq \tau \leq n+1$. Suppose there is no postacceleration, so that $A = d\theta/d\tau$ vanishes on this interval. Then θ is constant on this interval, say

 $\theta(\tau) = k$, a constant, for $\tau_f = n \le \tau \le n+1$. Substituting these facts into (28) gives

$$0 = 0 + \sinh(\theta(\tau - 1) - k) \quad \text{for} \quad n \le \tau \le n + 1 \quad ,$$

which may be restated as:

$$\theta(\tau) \equiv k \quad \text{for} \quad n-1 \leq \tau \leq n$$

Next consider equation (28) on the interval $n-1 \le \tau \le n$. Using the above observation that $\theta(\tau) \equiv k$ on this interval, which implies that $A \equiv 0$ on this interval, we see that

$$E(\tau) = -\sinh(\theta(\tau - 1) - k) \quad \text{for} \quad n - 1 \le \tau \le n \quad . \tag{29}$$

This says that, given the values of $\theta(\tau)$ for $n-2 \leq \tau \leq n-1$ (i.e., the values of $\theta(\tau-1)$ for $n-1 \leq \tau \leq n$), there is a unique force function $E(\tau)$ on $n-1 \leq \tau \leq n$ which will eliminate postacceleration. Put differently, for any other force function on $n-1 \leq \tau \leq n$, postacceleration must occur.

To complete the argument, suppose there is no preacceleration, i.e., $A(\tau) \equiv 0$ for $\tau < 0$. Then for some constant θ_0 , $\theta(\tau) = \theta_0$ for $\tau < 0$. Solve the DD1 equation (14) forward in time starting with this generalized initial condition, $\theta(\tau) \equiv \theta_0$ for $-1 \leq \tau \leq 0$. This will produce a unique solution $\theta(\tau)$ on the interval $0 \leq \tau \leq n - 1$, and this solution depends only on the values of $E(\tau)$ for $0 \leq \tau \leq n - 1$. Substituting this solution in equation (29) shows that postacceleration must occur for any force function $E(\tau)$ on $n-1 \leq \tau \leq n$ which does *not* satisfy equation (29), i.e., for almost all force functions.

The above argument becomes much simpler for the case n = 1, corresponding to a force applied only for a time interval of length 1, i.e., a time interval the length of the delay parameter in the DD equation. In Rohrlich's motivation of the DD equation, this time interval is the time light required to traverse the spherical particle's diameter, so it is probably too small to have any observational interest, but an examination of this special case is a quick way for the reader to become convinced that it is mathematically unreasonable to expect to simultaneously eliminate both preacceleration and postacceleration in general.

The simpler argument for n = 1 goes as follows. Observe as before that $\theta(\tau)$ must be constant for $n - 1 \leq \theta \leq n$, i.e., $\theta(\tau) = k$ for $0 \leq \theta \leq 1$, and in particular, $\theta(0) = k$. If there is no preacceleration, then $\theta(\tau)$ must be constant for $\tau < 0$, say $\theta(\tau) \equiv \theta_0$, for $\tau \leq 0$. In particular, $\theta_0 = \theta(0) = k$. Hence on [0, 1],

$$\theta(\tau - 1) = \theta_0 = k \quad \text{for} \quad 0 \le \tau \le 1$$

Substitute this in (29) to obtain

$$E(\tau) = -\sinh(k-k) \equiv 0 \text{ for } 0 \le \tau \le 1$$

This says that for an force applied no longer than the time delay, either precceleration or postacceleration *must* occur except in the trivial case of identically vanishing force.

5 Forward asymptotics for eventually vanishing force

This section proves that solutions of the DD1 equation (14) for eventually vanishing force have proper acceleration asymptotic to zero and asymptotically constant velocity in the future:

$$\lim_{\tau \to \infty} A(\tau) = 0 \quad ; \quad \lim_{\tau \to \infty} \theta(\tau) = \text{constant} \quad . \tag{30}$$

Thus replacing the Lorentz-Dirac equation by the DD equation does eliminate the undesirable "runaway" solutions of the former for the special case of motion in one space dimension. Whether this is also true for motion in three space dimensions is unknown.

Suppose that E eventually vanishes, and choose the origin of time so that $E(\tau) = 0$ for $\tau \ge 0$. Then for $0 \le \tau \le 1$, A satisfies

$$m\frac{dA}{d\tau} = \cosh(\theta(\tau - 1) - \theta(\tau))(A(\tau - 1) - A(\tau))$$
(31)

We shall consider the related equation

$$\frac{dh}{d\tau} = \phi(\tau)(g(\tau) - h(\tau)), \ 0 \le \tau \le 1, \text{ with initial condition } h(0) = g(1).$$
(32)

Here ϕ will be a given continuous function on [0, 1], and g another function with the same domain. We will think of ϕ as fixed until further notice, and of (32) as defining a mapping which assigns to each g another function h, namely the unique solution of (32). We will show that this mapping is a strict contraction relative to a certain Banach space norm.

Let C[0,1] denote the real Banach space of all continuous real-valued functions on [0,1] with the supremum norm $\|\cdot\|_{\infty}$: for $g \in C[0,1]$,

$$||g||_{\infty} := \max\{ |g(\tau)| \mid 0 \le \tau \le 1 \}$$

For $g \in C[0,1]$, let $M(g) := M_{[0,1]}(g)$ and $m(g) := m_{[0,1]}(g)$ denote the maximum and minimum of g.

Let *B* denote the Banach space which is the quotient of C[0, 1] by the onedimensional subspace of constant functions. For $g \in C[0, 1]$, let \tilde{g} denote its image in *B* under the quotient map. Denoting by **1** the function constantly equal to 1, and by **R** the real field, the quotient norm $\|\cdot\|$ on *B* is:

$$\|\tilde{g}\| := \inf\{ \|g + \alpha \mathbf{1}\|_{\infty} \mid \alpha \in \mathbf{R} \} = \frac{1}{2} [M(g) - m(g)] \quad .$$
(33)

The first equality is the definition of the quotient norm, and the last is a simple exercise.

Let ϕ be a given function, and consider the linear mapping $Q_{\phi} : C[0,1] \to C[0,1]$ defined as follows. For any $g \in C[0,1]$,

$$Q_{\phi}g := h \quad , \tag{34}$$

where h is the unique solution of (32).

Define $\tilde{Q}_{\phi}: B \to B$ to be the analog of Q_{ϕ} on the quotient space B: for all $g \in C[0, 1]$,

$$\tilde{Q}_{\phi}(\tilde{g}) := \widetilde{Q_{\phi}g} \tag{35}$$

We see that \tilde{Q}_{ϕ} is well-defined by noting that for any $\alpha \in \mathbf{R}$,

$$\frac{d(h+\alpha\mathbf{1})}{d\tau} = \frac{dh}{d\tau} = \phi(\tau)[g(\tau) - h(\tau)] = \phi(\tau)[(g(\tau) + \alpha) - (h(\tau) + \alpha)] \quad ,$$

and also $(h + \alpha \mathbf{1})(0) = h(0) + \alpha = g(1) + \alpha = (g + \alpha \mathbf{1})(1)$. Thus if we alter g by adding a constant α to it, then we also alter the solution of (32) by the same additive constant, so that the h in (35) depends only on the equivalence class of g in C[0, 1] modulo the constants.

Next we show that for any non-negative ϕ , the mapping $\hat{Q}_{\phi} : B \to B$ is a strict contraction. Actually, we'll need the following stronger fact giving a uniform bound on $\|\tilde{Q}_{\phi}\|$ for any bounded set of non-negative ϕ :

Lemma 4 For each non-negative $\phi \in C[0,1]$, $\|\tilde{Q}_{\phi}\| < 1$. Moreover, given any constant k_0 , there is a constant k < 1 such that for all non-negative functions ϕ with $\|\phi\| \leq k_0$, and for all $g \in C[0,1]$,

$$\|\tilde{Q}_{\phi}\tilde{g}\| \le k\|\tilde{g}\| \quad . \tag{36}$$

Proof: For each non-negative function $\phi \in C[0, 1]$, define

$$\psi(x) := \int_0^\tau \phi(s) \, ds$$

Note that since ϕ is non-negative, ψ is non-decreasing.

Then for any $g \in C[0,1]$, $(Q_{\phi}g)(\tau) = h(\tau)$ with h the solution to (32):

$$h(\tau) = e^{-\psi(\tau)}g(1) + e^{-\psi(\tau)} \int_0^\tau e^{\psi(s)}\phi(s)g(s) \, ds$$

 \mathbf{SO}

$$\begin{aligned} h(\tau) &\leq e^{-\psi(\tau)}g(1) + M(g)e^{-\psi(\tau)} \int_0^\tau e^{\psi(s)} \frac{d\psi(s)}{ds} \, ds \\ &= e^{-\psi(\tau)}g(1) + M(g)(1 - e^{-\psi(\tau)}) \\ &= M(g) + (g(1) - M(g))e^{-\psi(\tau)} \ . \end{aligned}$$

Since $g(1) - M(g) \leq 0$ and ψ is non-decreasing,

$$M(h) \leq M(g) + [g(1) - M(g)]e^{-\psi(1)}$$

= $M(g)(1 - e^{\psi(1)}) + g(1)e^{-\psi(1)}$ (37)

Similarly,

$$h(\tau) \ge e^{-\psi(\tau)}g(1) + m(g)(1 - e^{-\psi(\tau)})$$

and since $g(1) - m(g) \ge 0$ and ψ is non-decreasing,

$$m(h) \geq m(g) + [g(1) - m(g)]e^{-\psi(1)} = m(g)(1 - e^{-\psi(1)}) + g(1)e^{-\psi(1)} .$$
(38)

Subtracting the inequalities (37) and (38) shows that for each non-negative ϕ , \tilde{Q}_{ϕ} is a strict contraction:

$$\begin{aligned} \|\tilde{Q}_{\phi}(\tilde{g})\| &= [M(h) - m(h)]/2\\ &\leq (1 - e^{-\psi(1)})[M(g) - m(g)]/2 = (1 - e^{-\psi(1)})\|\tilde{g}\|. \end{aligned} (39)$$

Finally, the uniformity condition follows from (39) with $k := 1 - e^{-k_0}$, since $\|\phi\| \le k_0$ implies $\psi(1) \le k_0$ and hence $1 - e^{-\psi(1)} \le 1 - e^{-k_0}$.

Now we prove the main result of this section:

Theorem 5 Suppose that the force $\tau \mapsto E(\tau)$ in the DD1 equation (14) eventually vanishes, meaning that there exists a proper time τ_0 such that $E(\tau) = 0$ for $\tau \geq \tau_0$. Then there exists a constant θ_{∞} such that

$$\lim_{\tau \to \infty} A(\tau) = 0 \quad and \quad \lim_{\tau \to \infty} \theta(\tau) = \theta_{\infty} \quad .$$
(40)

Proof: We choose the origin of proper time so that $E(\tau) = 0$ for $\tau \ge 0$. In solving the acceleration equation (16), we assume that the solution A is already obtained for $\tau \le 0$. Then the solution on [0, 1] is obtained by applying an opertor Q_{ϕ} to the restriction A|[-1, 0] of A to [-1, 0], translated one unit right.

More precisely, define an operator T on C[0, 1] to be right translation by one unit: $(Tf)(\tau) := f(\tau - 1)$ for $f \in C[0, 1]$. Let $\phi(\tau) := \cosh(\theta(\tau - 1) - \theta(\tau))$, $0 \le \tau \le 1$. Then

for
$$0 \le \tau \le 1$$
, $A(\tau) = Q_{\phi}T(A|[-1,0])(\tau)$

Though we don't know $\theta(\tau)$ for $0 \le \tau \le 1$, we do know from Proposition 1 that it is bounded above and below by bounds no worse than on the preceding interval $-1 \le \tau \le 0$. Hence we have the *a priori* bound

$$\|\phi\|_{\infty} \le k_0 := \cosh(|M(\theta|[-1,0])| + |m(\theta|[-1,0])|)$$

The solution $A(\tau)$ for $1 \leq \tau \leq 2$ is similarly obtained, except that the Q_{ϕ} is different because the ϕ is different. However, Proposition 1 shows that we have the same *a priori* bound on the new $\|\phi\|_{\infty}$. After *n* applications of Lemma 4, we find that

$$(M(A|[n-1,n]) - m(A|[n-1,n]))/2 = ||A|[n-1,n]|| \le k^n ||A|[-1,0]|| \quad .$$

This implies that A|[n-1,n], becomes asymptotically constant as n becomes large. The constant must be zero because $A = d\theta/d\tau$, and the previous section showed that the rapidity θ is bounded on $[0, \infty]$. Hence

$$\lim_{\tau \to \infty} A(\tau) = 0 \quad . \tag{41}$$

Finally, we show that the rapidity (hence the velocity) becomes asymptotically constant in the future: for some constant θ_{∞} ,

$$\lim_{\tau \to \infty} \theta(\tau) = \theta_{\infty} \quad . \tag{42}$$

From the last section, $M_{[n,n+1]}(\theta)$ is a nonincreasing function of the positive integer n, and $m_{[n,n+1]}(\theta)$ is nondecreasing. This implies that if (42) were false, there would exist a positive constant δ with $M_{[n,n+1]}(\theta) - m_{[n,n+1]}(\theta) \geq \delta$ for all $n \geq 0$. Choose points τ_n^+ and τ_n^- in [n, n+1] with $\theta(\tau_n^+) = M_{[n,n+1]}(\theta)$ and $\theta(\tau_n^-) = m_{[n,n+1]}(\theta)$. Suppose $\tau_n^- \leq \tau_n^+$. Applying the Mean Value Theorem to the interval $[\tau_n^-, \tau_n^+]$ yields a point τ_n between τ_n^- and τ_n^+ with

$$A(\tau_n) = \frac{\theta(\tau_n^+) - \theta(\tau_n^-)}{\tau_n^+ - \tau_n^-} \ge \frac{M_{[n,n+1]}(\theta) - m_{[n,n+1]}(\theta)}{1} \ge \delta \quad .$$
(43)

Similarly, if $\tau_n^+ < \tau_n^-$, then $A(\tau_n) \leq -\delta$ for some $\tau_n \in [n, n+1]$; both this and (43) contradict the previously established fact that $\lim_{\tau \to \infty} A(\tau) = 0$.

6 Summary and assessment

The above proofs were given in the context of the normalized DD equation (7) for motion in one space dimension and its equivalent formulation, the DD1 equation (14). Translating them into information about the original DD equation (6) discussed in (Rohrlich, 1997, 1999) yields the following observations, in which the motion is assumed restricted to one spatial dimension unless otherwise specified.

1. An appropriate condition to guarantee a unique solution is a specification of the rapidity $\theta(\tau)$ on some closed interval $[\alpha, \alpha + \tau_1]$, subject to the consistency condition (18) (written there for $\alpha := 0$ and $\tau_1 = 1$). We refer to this specification as a "generalized initial condition".

There exists a unique C^1 solution $\theta(\cdot)$ of (14) satisfying any C^1 generalized initial condition, and hence a unique four-velocity $u(\tau) = (\cosh \theta(\tau), \sinh \theta(\tau))$ and worldline $\tau \mapsto z(\tau)$ with $dz/d\tau = u(\tau)$ (by integration, with arbitrary specification of z(0)).

2. For the case of a force $\tau \mapsto E(\tau)$ applied for only a finite time (i.e., compactly supported force), there exist solutions which do not vanish asymptotically in the distant past. Such solutions seem unphysical. For simplicity of language, we will call them "past runaway" solutions even though we do not know that the acceleration becomes unbounded in the infinite past.

However, for compactly supported force, there exists a unique choice of generalized initial condition which eliminates these past runaway solutions. This seems the physically relevant choice of initial condition.

- 3. For a force not compactly supported, it has not been proved that there is an appropriate choice of generalized initial condition to eliminate past runaways. Indeed, it seems unclear what such a choice might be. If the DD equation is to be used to predict the motion of actual particles, some definite procedure for specifying the generalized initial condition would be necessary.
- 4. For a force E which eventually vanishes (i.e., for some τ_0 , $E(\tau) = 0$ for $\tau \geq \tau_0$), the acceleration does converge to zero in the future, and the velocity becomes asymptotically constant. Thus for the DD equation for motion in one space dimension, analogs of the pathological "future runaway" solutions of the Lorentz-Dirac equation do not occur. It is unknown whether future runaways are also impossible for general three-dimensional motion obeying the DD equation.
- 5. For compactly supported force, there exist "preaccelerative" solutions (defined as solutions with nonzero acceleration *before* the force is applied). Such solutions seem unphysical because they violate "causality". However, the same generalized initial condition which rules out past runaways also rules out these undesirable preaccelerative solutions.
- 6. For compactly supported nonzero force and generalized initial condition chosen to rule out preacceleration, the resulting solution necessarily exhibits the unusual property which we call "postacceleration": the acceleration persists after the force is removed, and in fact into the infinite future. Some may consider this strange behavior sufficiently unphysical to rule out the DD equation. Others may welcome the unusual prediction as a potential physical test which if verified, would constitute striking evidence for the equation.

Given that known motivations of the DD equation are not fundamental (e.g., employ uncontrolled approximations obtained by ignoring nonlinear terms), and given the uncertainties noted above about the mathematical appropriateness of the equation, it seems that more work would be needed to convert it from a speculative proposal to an accepted physical principle.

It is often said that a resolution of the logical problems of classical electrodynamics must come from more fundamental physical principles of quantum mechanics. This may indeed turn out to be true, but it is not the only possibility. It would seem strange if a logically consistent and physically sensible classical electrodynamics were inherently impossible. Although nearly all of physics rests in principle on quantum mechanics, there do exist consistent and sensible classical theories in many areas. The present theory of classical charged particles cannot be considered sensible.

It is interesting that the Lorentz-Dirac equation has survived as the principal candidate for a classical equation of motion despite predictions so bizarre (e.g., Eliezer, 1943, Parrott, 2002) that no one will admit to believing them. The reason for the survival may be the fundamental nature of Dirac's derivation of the equation. If one accepts mass renormalization (admittedly controversial), then one can convincingly obtain the Lorentz-Dirac equation from the principle of conservation of energy-momentum with no approximations whatever. One does not lightly discard such mathematically tight arguments.

The motivations of many other proposed equations of motion are aesthetically less pleasing. Most consist of more or less *ad hoc* modifications of the Lorentz-Dirac equation which do not *obviously* lead to its bizarre predictions. (This is not the same as saying that the modifications obviously *do not* lead to similarly bizarre predictions!)

Assuming that one is willing to believe in postacceleration, the DD equation seems physically possible, but not physically compelling. It is not clear in what sense, if any, solutions of the DD equation conserve energy-momentum. It would be desirable to find a convincing, mathematically rigorous way to relate the DD equation to the principle of conservation of energy-momentum.

7 Appendix 1

When the body of this paper was written, I was unaware of seminal work of Ryabov (Ryabov, 1960, 1961, 1963, 1965) on delay differential equations, subsequently refined and extended by (Driver, 1968). This work does not directly apply to the DD equation because this equation does not satisfy Driver's hypotheses. However, it has the same structure as equations considered by Driver.

If sinh were a bounded function, then Driver's results *would* apply to Rohrlich's equation (14). Thus we can hope that conclusions similar to Driver's might apply to the DD equation. If so, they would shed light on some of the problems mentioned in the Summary and Assessment section.

This appendix outlines Driver's conclusions in the context of the DD equation. It is mathematically based on the excellent semi-expository paper (Driver, 1968), but the presentation follows the introductory sections of (Chicone, 2003), which is based on Driver's work. The latter presents a point of view particularly congenial to a discussion of the DD equation.

We shall be concerned with delay-differential equations of the form (15), but for easy exposition we need to reinsert the positive delay parameter τ_1 (which was set to 1 in (15) by appropriate choice of units):

$$\frac{d\lambda}{d\tau} = \Phi(\tau, \lambda(\tau), \lambda(\tau - \tau_1)) \quad . \tag{44}$$

In equation (15), the function λ was a real-valued function, but the case in which λ takes values in \mathbb{R}^n is no more difficult to discuss, and the results that we shall present apply also to this case. Therefore, we assume that $\lambda(\tau) \in \mathbb{R}^n$.

The specialization of Driver's work of interest to us applies to equations of this form with Φ continuous and bounded. The hypothesis that Φ be continuous and bounded makes the results easy to state. It can be relaxed, but not so far as to admit the Rohrlich equation. More precisely, the hypothesis that Φ be bounded can be relaxed to an assumption that Φ satisfy a certain Lipschitz condition.

A very important hypothesis for the results of Ryabov and Driver to be described is that τ_1 be sufficiently small. That is why we needed to reinsert the delay as an explicit parameter in (44).

Previously we normalized τ_1 to 1 by appropriate choice of units, and the reader may be puzzled as to why we can't maintain the same normalization here. We could, but it's not convenient. The reason is that normalizing τ_1 to 1 correspondingly changes the Lipschitz constant, so that the restriction that τ_1 be sufficiently small is replaced in general by a restriction on the Lipshitz constant, or by a restriction on the bound of Φ under our simpler hypothesis that Φ be continuous and bounded. From a physical point of view, it is more natural to phrase the hypothesis for the Ryabov/Driver results in terms of a restriction on the time delay instead of a restriction on the Lipshitz condition or the bound of Φ .

A priori, a mathematically appropriate initial condition to specify a unique solution for (44) is a specification of the function $\lambda(\cdot)$ on an interval $[0, \tau_1)$ subject to the consistency condition $d\lambda/dt(\tau_1) = \Phi(0, \lambda(\tau_1), \lambda(0))$. Thus the solution space of (44) is infinite dimensional.

For the DD1 equation (14) (corresponding to n = 1 in (44)), the physical expectation is that there should be precisely one physically relevant solution for each ordinary initial condition $\lambda(0) = \lambda_0 \in \mathbb{R}^1$. Thus the space of mathematical solutions of (44) is too large—one needs to find a condition which will reduce the infinite-dimensional solution space to a one-dimensional space. The Ryabov/Driver theory suggests such a condition, as we shall now discuss.

A solution λ of (44) is called a *special solution* if

$$\sup_{-\infty < \tau < \infty} e^{-|\tau|/\tau_1} \|\lambda(\tau)\| < \infty \quad . \tag{45}$$

Thus a special solution is one which does not increase at too fast an exponential rate. In particular, all solutions which are *not* special are "runaway" (either in the past or the future), and therefore presumably unphysical for physically possible forces.

Driver's results imply that for every $\lambda_0 \in \mathbb{R}^n$, there exists a unique special solution (to (44) with Φ continuous and bounded) satisfying $\lambda(0) = \lambda_0$. Thus the space of special solutions is a manifold of dimension n, coordinatized by the map $\lambda \mapsto \lambda(0)$.

In addition, Driver showed that this manifold is an attractor in the sense that *every* solution approaches (is "attracted" to) a special solution (exponentially fast) in the infinite future. Thus the manifold of special solutions models the long-term behavior of all solutions.

The DD equation (7) in *m*-dimensional space ((m + 1)-dimensional spacetime), can be reformulated as an equivalent equation of form (44) with n = m. Here we are primarily interested in the case m = 3, and secondarily in m = 1. The reason that (7) with m = 3 is not immediately of the form (44) is that a four-velocity u must satisfy the auxiliary condition $u^{\alpha}u_{\alpha} = 1$. Taking this into account reduces it to an equivalent equation of form (44) with m = 3; we omit the details. Although the corresponding Φ on its right side is not bounded, we can hope that the conclusions of the Ryabov/Driver theory might still apply.

Suppose that the force in the DD equation (7) is continuous and compactly supported (hence bounded), and consider the equivalent equation of form (44) with m = 3. Then all physically relevant solutions should be bounded, hence special (because all non-special solutions are "runaway", as noted above). If the conclusions of the Ryabov-Driver theory extend to this more general situation, then the space of physically relevant solutions has dimension no greater than m.

In general, special solutions can increase exponentially fast, so we cannot immediately identify special solutions with "physical" solutions. However, it is unknown whether exponential growth of special solutions can occur for the DD equation in four-dimensional spacetime.

If there were exponentially increasing special solutions to the DD equation in four-dimensional spacetime with compactly supported force, this would seem to definitively rule it out as a physically realistic equation of motion. However, it is an attractive conjecture that in this situation, all special solutions must be bounded. If so, then the space of physically relevant solutions coincides with the space of special solutions, and it has the physically expected dimension 3. Similar remarks apply to the DD equation in spacetimes of other dimensions.

More generally, it is attractive to speculate that under reasonable hypotheses on a force not compactly supported (e.g., an exponentially decreasing force), all special solutions might be bounded. If this could be proved, it might be considered as a solution in principle to the problem posed in comment 3 of the Summary and Assessment section: Is there a generalized initial condition which will eliminate runaways in this situation? That is, we could identify "physical" (e.g., non-runaway) solutions with special solutions. However, since closed form special solutions are typically hard to come by, the practical problem of identifying the special solution in some constructive way would remain.

The only thing which prevents us from directly applying the insights of the Ryabov/Driver theory to the DD1 equation (14) is that the function sinh does not satisfy the required Lipschitz condition. But it seems possible that this may be merely a technical matter, and it seems reasonable to hope that the Ryabov/Driver theory might be extended to apply to this equation. If so, Theorem 5's conclusion that the rapidity $\theta(\tau)$ converges to a constant as $\tau \to \infty$ illustrates how the manifold of special solutions models the long-term behavior of the system. If every special solution converges to a constant in the future (as the theorem establishes for eventually vanishing force), then *every* solution must similarly converge to a constant (as the theorem also establishes).

However, even if the Ryabov/Driver theory could be extended to cover the DD equation (7), there would be another difficulty which could be more fundamental. This difficulty is that even for delay equations (44) to which the Ryabov/Driver theory does apply, it only applies for sufficiently small τ_1 .⁴

⁴See, e.g., (Chicone, 2003) for an explicit bound $\tau_1 < \delta$ which guarantees applicability

Unfortunately, in our application, it is questionable to assume that τ_1 is arbitrarily small. Indeed, a critic of an earlier version of the present work raised the following interesting objection.

In Rohrlich's formulation, the classical charged particle is treated as a sphere of nonzero radius and the delay τ_1 is twice that radius (the time for light to traverse the sphere). If the radius is too small, then the classical electrostatic energy of such a sphere will exceed the typical energy required for the quantum effect of pair production, moving the problem out of the domain of classical electrodynamics and into the domain of quantum electrodynamics. But the DD equation is proposed solely as a classical equation of motion.

Put differently, the considerations leading to the DD equation are not claimed to apply to arbitrarily small delays. The critic mistakenly thought that the present work assumed Caldirola's value of $\tau_1 = 4q^2/(3mc^3) \approx 1.2 \times 10^{-23}$ sec, which he believed would be far less than could be the radius of any classical charged particle.

Therefore, any proposed application of the Ryabov/Driver theory to the DD equation within the framework of motivation via a nonzero particle radius should address the question of whether the delay (equivalently, the particle's radius) is small enough for an extended Ryabov/Driver theory to apply.

8 Appendix 2

This appendix gives the details of the equivalence of

$$m_1 \frac{du^i}{d\tau} = f^i(\tau) + m_2 [u^i(\tau - \tau_1) - u^\alpha(\tau - \tau_1)u_\alpha(\tau)u^i(\tau)] \quad , \tag{6}$$

and

$$m\frac{d\bar{u}^{i}}{d\bar{\tau}} = \bar{f}^{i} + [\bar{u}^{i}(\bar{\tau}-1) - \bar{u}^{\alpha}(\bar{\tau}-1)\bar{u}_{\alpha}(\bar{\tau})\bar{u}^{i}(\bar{\tau})] \quad .$$
(10)

We'll show that these equations are equivalent if the barred quantities are defined by:

$$\bar{u}(s) := \tau_1 u(\tau_1 s) \text{ for all real numbers } s,$$
(46)

$$\bar{f}(s) := \tau_1 f(\tau_1 s) / m_2$$
 for all real numbers s, and (47)

$$m := m_1/(m_2\tau_1)$$
 . (48)

Call the time unit with respect to which (6) is written the "old time unit". First we change to a new time unit which equals τ_1 old time units. If the old metric tensor is denoted $\eta := \text{diag}(1, -1, -1, -1)$, this can be accomplished by introducing a new metric $\bar{\eta} := \tau_1^{-2} \eta$. This changes the old proper time τ to a new proper time

$$\bar{\tau} = \tau_1^{-1} \tau$$
 , $\tau = \tau_1 \bar{\tau}$.

of the Ryabov/Driver results. Unfortunately for our application, the bound δ is inversely proportional to the Lipschitz constant, which is formally infinite for sinh (so that δ is formally zero).

A worldline $\tau \mapsto z(\tau) \in R^4$ parametrized by old proper time τ corresponds to a "new" worldline $\bar{\tau} \mapsto \bar{z}(\bar{\tau})$ parametrized by new proper time $\bar{\tau}$, with $\bar{z}(\bar{\tau}) = z(\tau)$, i.e., for any real number s,

$$\bar{z}(s) = z(\tau_1 s)$$

The "new" worldline consists of the same set of points as the old worldline, but the parametrization is different.

The four-velocity $\bar{u}(\bar{\tau})$ of the new worldline is related to the old four-velocity $u(\tau)$ by:

$$\bar{u}(\bar{\tau}) := \frac{d\bar{z}}{d\bar{\tau}} = \tau_1 u(\tau), \quad \text{equivalently}, \quad u(\tau) = \tau_1^{-1} \bar{u}(\bar{\tau})$$

Thus \bar{u} and u are related by the simple transformations

$$\bar{u}(s) = \tau_1 u(\tau_1 s), \quad \text{and} \quad u(s) = \tau_1^{-1} \bar{u}(\tau_1^{-1} s), \quad \text{for all real numbers } s.$$
 (49)

This gives a way to translate any statement about \bar{u} into a corresponding statement about u, and vice versa. Similarly,

$$\frac{d\bar{u}}{d\bar{\tau}} = \tau_1^2 \frac{du}{d\tau}, \quad \text{equivalently}, \quad \frac{du}{d\tau} = \tau_1^{-2} \frac{d\bar{u}}{d\bar{\tau}}.$$
(50)

Next, divide (6) by m_2 , and substitute the above relations (49) and (50), obtaining

$$\frac{m_1}{m_2}\tau_1^{-2}\frac{d\bar{u}^i}{d\bar{\tau}} = \frac{f^i(\tau)}{m_2} + \tau_1^{-1}[\bar{u}^i(\bar{\tau}-1) - \bar{u}^\alpha(\bar{\tau}-1)\bar{u}_\alpha(\bar{\tau})\bar{u}^i(\bar{\tau})] \quad .$$
(51)

Now inspection shows that after multiplication of (51) by τ_1 , (10) will result if we define \bar{u}, \bar{f} , and m as in equations (46), (47), and (48) above.

9 Appendix 3: Comment on the analysis of (Moniz and Sharp, 1977) of a non-relativistic version of the DD equation

(Rohrlich, 1997) states the following, concerning the DD equation and a non-relativistic simplification of it which will be discussed below:

"Returning to the overview of classical charged particle dynamics, one can summarize the present situation as very satisfactory: for a charged sphere there now exist equations of motion both relativistically [this refers to the DD equation (6)] and nonrelativistically that make sense and that are free of the problems that have plagued the theory for most of this century; these equations have no unphysical solution, no runaways, and no preaccelerations." This resolves into the following claims:

Claim 1: The DD equation has no preaccelerative solutions;

Claim 2: The DD equation has no solutions which are runaway (in the future);

Claim 3: The DD equation has no "unphysical solution".

The analysis of this paper shows that all of these claims are at best optimistic and at worst false.

We showed that Claim 1 is false as stated. However, we also noted that it can be reformulated (by adjoining appropriate generalized initial conditions) so as to become true for the special case of an field applied for a finite time. It is unknown whether the claim can be repaired for arbitrary fields, as discussed in Section 3.

We showed that Claim 2 is true for a particle moving in one space dimension under a force which eventually vanishes; it is unknown whether it is true for general three-dimensional motion, as discussed in Appendix 1. Since the proof we gave for one dimension was nonroutine and special to that dimension, we suspect that new ideas will be required for a three-dimensional proof.

The truth of Claim 3 depends on what one means by "unphysical" solutions, but if one considers both preacceleration and postacceleration "unphysical", then we showed that this claim cannot be repaired.

The only evidence for any of these claims offered by (Rohrlich, 1997) or (Rohrlich, 1999) is an analysis by (Moniz and Sharp, 1977) of a non-relativistic version of the DD equation. Since the non-relativistic version is an entirely different equation, even if the claims were true for the non-relativistic version, they would not imply corresponding claims for the DD equation. However, a careful reading of (Moniz and Sharp, 1977) reveals that their analysis does not even prove the claims for their nonrelativistic equation.

The following passage from (Moniz and Sharp, 1977) has sometimes been interpreted as making Claims 1 and 2 for their nonrelativistic version of the DD equation:

[From the last paragraph of Section II, p. 2856] "Summarizing, we have found that including the effects of radiation reaction on a charged spherical shell results neither in runaway behavior nor in preacceleration if the charge radius of the shell $L > c\tau \dots$ "

This quote is from the part of the paper which treats only classical charged particles (as opposed to the quantum-mechanical treatment of later sections).

It is not clear whether Moniz and Sharp intended to assert that *no* solution can be either runaway or preaccelerative (i.e., Claims 1 and 2), or that given ordinary initial conditions specifying the position and velocity at a given time, *there exists* a solution satisfying these conditions and which is neither runaway nor preaccelerative. The aim of the present appendix is to dispel the confusion over these various claims by analyzing precisely what (Moniz and Sharp, 1977) does prove. We shall see that what they actually show is closer to the latter than the former.

The condition $L > c\tau$ corresponds in our notation to the condition that our delay parameter τ_1 must not be too small. More specifically, the condition $L > c\tau$ translates into a condition

$$\tau_1 > \delta \quad , \tag{52}$$

where δ is a certain positive parameter whose exact value will not be important to us. Henceforth we assume this condition. (Incidentally, for other values of the parameter τ_1 , Moniz and Sharp do find unphysical solutions, either future runaway or oscillatory.)

Their nonrelativistic version of the DD equation is:

$$\frac{d\mathbf{u}}{dt} = \mathbf{h}(t) - b[\mathbf{u}(t - \tau_1) - \mathbf{u}(t)] \quad , \tag{53}$$

where $\mathbf{u}(t)$ represents the particle's three-dimensional velocity at time t, \mathbf{h} is a force-like term (a three-dimensional force divided by certain constants), and b is a constant. This is their equation (2.10) on p. 2853 written in notation closer to ours. It can be obtained from the space part of the DD equation (6) or (7) by neglecting terms of quadratic or higher order in the velocity.

It is clearly much simpler than the DD equation. In particular, it is what one might call a "linear" delay-differential equation, and this is critical to their analysis via Fourier transforms.

Moniz and Sharp's analysis of the possibility of solutions which are runaway in the future assumes that \mathbf{h} eventually vanishes, so that for sufficiently large times, it may be dropped from the right side of (53), obtaining

$$\frac{d\mathbf{u}}{dt} = b[\mathbf{u}(t-\tau_1) - \mathbf{u}(t)] \quad .$$
(54)

This equation admits exponential solutions

$$\mathbf{u}(t) := \mathbf{u}_0 e^{\alpha t} \quad , \tag{55}$$

with \mathbf{u}_0 a constant vector and α a complex constant satisfying $\alpha = b[e^{-\alpha\tau_1} - 1]$. Moniz and Sharp then show that for τ_1 satisfying (52), the real part of α is negative, so that these exponential solutions decay to 0 as t becomes large.

That is all they prove. But this only proves that no solutions of the simple *exponential* form (55) can be runaway in the future, which is not the same thing as proving that *no* solution can be runaway in the future. There are many solutions to (54) which are not exponential—indeed, the space of all solutions is infinite dimensional (as noted in our Section 3 and Appendix 1), whereas the space of exponential solutions only has dimension 3. Thus Moniz and Sharp's analysis proves neither that no solution of their equation (53) is runaway (Claim 2) nor that there exists a non-runaway solution satisfying an arbitrary initial condition $\mathbf{u}(t_0) = \mathbf{u}_0$.

(Moniz and Sharp, 1977) does give a prescription for writing down a formal Fourier transform of a solution of (53) for each forcing function \mathbf{h} . (They do not address the question of whether the formal expression produced, which contains singularities, actually is the distributional Fourier transform of a solution.) This prescription produces solutions which are not preaccelerative (in the precise sense defined in Section 3). Thus they show (to the standard of rigor common in physics journals like *Physical Review*, not to mathematical standards) that there exist non-preaccelerative solutions.

However, their proof is incomplete in one important respect—they do not show that there exists a nonpreaccelerative solution satisfying an arbitrary initial condition $\mathbf{u}(t_0) = \mathbf{u}_0$. The nonpreaccelerative solution \mathbf{u} which they produce is *unique*, leaving no room to satisfy general (ordinary) initial conditions.

Nevertheless, it seems possible that this proof could be completed. Even if so, this would not prove Claim 1's assertion that *no* solutions are preaccelerative, but they may not have intended to assert Claim 1.

Indeed, there is an simple, explicit counterexample to Claim 1 for the nonrelativistic DD equation (53). Consider the case in which the force vanishes identically, i.e., $\mathbf{h} \equiv 0$ in (53). For this case, *any* solution for which the acceleration $d\mathbf{u}/dt$ does not vanish identically is preaccelerative—the particle accelerates *before* the force is applied (because the force is never applied). So, the exponential solutions $u(t) = \mathbf{u}_0 e^{\alpha t}$ previously noted in (55) are preaccelerative when $\mathbf{u}_0 \neq 0$, in contradiction to Claim 1.

10 Appendix 4: Referees' reports; Part 1: History

This paper was first submitted to the *Journal of Mathematical Physics*. The first referee's report was superficial and demonstrably incorrect in some important respects, so I requested a second referee. The second referee recommended rejection on the sole grounds that the problem which both (Rohrlich, 1999) and my paper addresses (the problem of finding a sensible equation of motion for a classical charged particle) is not "relevant for physics".

The first referee only objected to the methods of the paper, not to its relevance. The second referee, who was presumably furnished the first referee's objections to the paper's methods, did not question the paper's methods.

I don't agree with the second referee's value judgment, but I can understand and respect it. His report was thoughtfully written and showed understanding of the paper. It is the only report of six (see below) which I can respect.

I had submitted the paper to *J. Math. Phys.* because of its mathematical content, some of which is at a higher level than is typically published by journals like *Physical Review*. After the rejection by *J. Math. Phys.*, I decided to submit it to *Phys. Rev. D* (PRD), which had published (Rohrlich, 1999). I thought (naively, it turned out), that having recently published Rohrlich's paper on the same topic, they would find it inconsistent to maintain that the problem which

both papers address was physically irrelevant.

The story to follow may not always reflect well on the standards of PRD, so before giving it, I'd like to express my appreciation for the efficiency of their online submission procedure. The mechanics of dealing with them is a pleasure, compared to most journals.

They are efficient and fast. Nobody likes a rejection, but an immediate rejection is infinitely more courteous than a rejection after several years of unanswered or inadequately answered correspondence. With other journals, the latter happens more often than it should. (I should also remark that J. *Math. Phys.* was also efficient.)

Initially, PRD sent the paper to an anonymous referee and to Professor Rohrlich as an identified referee, the latter because the paper commented on his. Professor Rohrlich thought that the paper's analysis was invalid because it replaces his original equation (6) with the equivalent equation (7). I wrote him privately spelling out the equivalence (essentially the same analysis as Appendix 2, which was added later), but he maintained his objection.

The anonymous referee submitted a short, superficial report giving the impression that the paper merely quoted "existing theorems", and objecting that its subject was "interesting for the foundations of physics but not for phenomenology". (I wonder what he'd say about string theory!)

I could understand this referee's value judgment if PRD were an experimental journal, but it seems inconsistent with its publication of Rohrlich's paper in 1999 and the continued publication of papers on the same topic since that time. I wondered if this referee had actually read the paper (there was no internal evidence of that in the review), and I wondered if he might be looking for excuses to avoid dealing with its mathematics.

Professor Rohrlich's objection to replacement of the equation (6) as he originally wrote it with the equivalent version (7) seemed so far out in left field that I felt I couldn't let it pass. So, I wrote the Editor explaining why I thought that Rohrlich had made a serious mistake, and was continuing to make it.

He sent the paper, including Appendix 2 spelling out the equivalence of (6) and (7), to a second anonymous referee (the "Third Referee", including Rohrlich). Third Referee reported that he "completely agree[s] with Professor Rohrlich's critique that 'results based on equation (7) are physically meaningless ...'".

I was flabbergasted, given that the issue in dispute is so elementary, and had been explained so carefully in Appendix 2. The report of Third Referee gave no indication that he had even read Appendix 2. Surely, if there were some error in the short Appendix 2, a conscientious referee should have pointed it out.

By this time, I was so disgusted that it was hard not to simply walk away from the matter, but I eventually decided that for completeness and closure, I should protest this latest incompetence. So, I wrote the Editor, D. Nordstrom, explaining that I believed that, however unlikely it might seem, Third Referee's report was also seriously in error.

I suggested the following way of resolving the matter to the satisfaction of all: submit the narrow issue of the equivalence of (6) and (7) to an identified referee

acceptable both to PRD and me. (I suggested Barry Simon, whom I have never met, but whose competence I respect, based on his published work.) I promised that if such a referee ruled that these two equations were not equivalent, I would immediately withdraw the paper and never submit another to PRD.

He didn't reply to that, but he did send the paper back to Third Referee, who reversed himself, writing:

"I have no doubt about the mathematical equivalence of equs. (6) and (7). (And I have read and confirmed the calculations in Appendix 2!)"

But this referee maintained his recommendation of rejection on the basis that (a) "the topic of a classical equation of motion does not belong to the most urgent questions in theoretical physics ...", and (b) the paper does not address the issue of the size of the postacceleration predicted by the DD equation.

11 Appendix 4, Part 2: All of the referees' reports, in full

All of the referees' reports follow, in full, accompanied by brief comments. I think they say a lot about current standards.

My experience has been that the literature in this field is shockingly unreliable something like 1/3 to 1/2 of all published papers contain errors sufficiently serious to potentially invalidate some of their main conclusions. I suspect that one reason is that few papers are carefully read by their referees.

That was certainly the experience with this paper. Five of the six reviews are either badly incorrect or unacceptably superficial. I suspect that such superficial or incompetent refereeing is more the rule than the exception. That hypothesis would certainly explain the unreliability of the literature.

Moreover, published errors are frequently perpetuated by repeated citations by authors who want to believe the erroneous conclusions but apparently don't feel it necessary to check the mathematics by which they were obtained. (Given the unreliability of the literature, a conscientious author should check the mathematics of any paper cited in an essential way.) An example is the citations by proponents of the DD equation (such as the first referee for *J. Math. Phys.*) of the flawed analysis by (Moniz and Sharp, 1977) of their non-relativistic analog of the DD equation (see Appendix 3).

The experience of this paper clearly shows how difficult it is to get errors corrected. It is so difficult, time-consuming, and unpleasant that most people wouldn't bother. It would be unlikely to help one's careeer, and it would probably harm it. Were I not retired (from academics, not from research), I'm not sure that I would have bothered, and I'm not sure I'll bother in the future.

I don't plan to submit this paper elsewhere. Few referees of physics journals are likely to be willing to read the mathematics to the level required to determine if it is a valid critique of the DD equation proposed by Rohrlich. Mathematical journals probably wouldn't touch it as written because of its physical content. Its mathematical interest is limited because it deals solely with a particular equation which has not achieved wide acceptance.

Given that this magnificent ArXiv is the final repository for this work, it seems appropriate to include the referees' reports so that anyone thinking of reading it carefully can judge for himself the validity of the objections which prevented its publication.

11.1 Reports of the referees for J. Math. Phys.

11.1.1 Report of Referee 1 for J. Math. Phys.

Referee's report "This is an interesting study of a delay-differential equation of the type that occurs for classical charged particles.

However, it is of little interest from the point of physics. The equation (6) that the author studies is NOT the equation given by Rohrlich in reference [11]. At best, it is a very special case of it in which the radius of the charged sphere, a, is identified with $c\tau_0/2$, where τ_0 is approximately 1.2×10^{-23} sec.⁵ Since equation (6) refers to a *classical* particle, *its radius must necessarily be much, much larger,* $a \gg \tau_0$.

The author is apparently not aware of the fact that the Rohrlich equation has been studied a long time ago, for example by Moniz and Sharp (Phys. Rev. **15**, 2850 (1977)). In particular, they studied the issues of run-away solutions and of preacceleration. And they found that for the physical situation, $a \gg \tau_0$, these two undesirable phenomena are *absent* (see Section II of their paper).⁶

The author's conclusions are therefore not generally applicable to the physical situation, where $a \gg \tau_0$. In fact, almost all of the author's summary and assessment (his Section 6) would be different for the physical situation as discussed for example in the excellent book by Yaghjian, *Relativistic Dynamics of a Charged Sphere* (Springer, 1992). He shows how the problems of run-away and preacceleration originate,⁷ and he actually derives the Rohrlich equation from first principles in the rest frame (Appendix

⁵Author's comment: In the version sent to this referee, the delay parameter in (6) was called τ_0 , and was originally introduced in the context of Caldirola's equation (2), where it was mentioned that Caldirola had assumed the approximate value 1.2×10^{-23} for it. The referee apparently assumed that the mathematical analysis would only apply to this value! (Of course, it applies to any value.) Because of this, in later versions the delay parameter was renamed τ_1 .

⁶Author's comment: The errors and questionable statements in the cited paper of Moniz and Sharp are analyzed in detail in Appendix 3 (which was not included in the version sent to this referee because the issue had never come up). And even if Moniz and Sharp's claims concerning nonexistence of runaways and preacceleration were entirely correct, they would not imply similar claims for the DD equation because Moniz and Sharp analyze a much simpler equation.

⁷Author's comment: I agree neither with this nor with similar statements in (Rohrlich, 1997). But a careful analysis would take many pages.

D), from which it can easily be generalized to an arbitrary inertial frame.

Thus while this paper is *mathematically* interesting, it has no value for physics. In fact, it may contribute to the confusion that already exists, and to which the author has unfortunately succumbed."

11.1.2 Report of Referee 2 for J. Math. Phys.

Referee's report I have examined this MS, as well as the report of the first referee, and the subsequent reply by the author.

The MS is concerned with the motion of a charged particle in an electromagnetic field within the framework of classical physics. I first make some remarks, inspired by what the autor [sic] says in the Introduction and in Sect. 6. It is obvious that no *comprehensive* classical theory of the interaction of charged particles with the electromagnetic field can be physically correct, in view of what is known today. Such a theory, which has actually never been presented, could be "correct" in the sense that it is mathematically consistent, but its formulation would be an intellectual exercise not relevant for physics. A physically correct theory should presumably be within the framework of quantum mechanics. Quantum electrodynamics is such a theory, with many spectacularly successful applications (although not without some problems). Although quantum mechanics is the basic framework of physics, there are many special situations in which a description within a classical theory is perfectly adequate for practical applications. Such a classical theory is "correct" if it accounts for physical phenomena within its domain of applicability, and we then expect that it can be derived (as an approximation) from quantum mechanics, within this domain. The simple Lorentz force equation, for the motion of a charged particle in a given external electromagnetic field, can be regarded as an example. It plays a role in plasma physics, and is important in the theory of cosmic rays. Charged particle optics is based on this equation, and it is used successfully by designers of accelerators and storage rings (with some refinements to take into accout [sic] energy loss by radiation).

The MS is concerned with *modifications* of the Lorentz force equation. After an introductory discussion of various equations of motion which have been suggested in the past, the author then studies a particular equation, Eq. (6), attributed to Rohrlich, in the special case of *one*-dimensional motion. This represents a drastic simplification, but it makes the problem tractable. His findings are summarized in Section 6, in which a number of unphysical features are also noted.

I am extremely skeptical about the kind of equations of motion considered in the MS, and the author actually seems to share in the skepticism, judging from what he says in Sect. 6. They are of an *ad hoc* nature, without *any* real physical motivation, and I cannot imagine any actual physical situation in which, say, Eq. (6) would be relevant. In discussing the motion of an extended object, say a "large" molecular ion, the internal structure (and dynamics) of the object would have to be considered (in some situations), and the description of the object as a charged sphere or ball would not make much physical sense. Many particles also have spins and magnetic moments, but the equations in the MS do not deal with such features. There is a considerable literature on classical equations of the kind considered in the MS, and the number of papers might convey the impression that the subject is "well established". I think, however, that one has to look critically at what these theories really try to accomplish, and whether they are actually relevant for physics.

I do not think that the MS is appropriate for the JMP because I do not think the discussion is relevant for physics. However, the purely *mathematical* discussion in Sects. 3-5 might be of interest to some people, and I suggest that the author considers [sic] submission to some journal on differential equations."

Author's comment: This is a thoughtful report, and I agree with much of what the referee says. However, I think he goes overboard in his criticism that equations of motion like the Lorentz-Dirac and DD equation do not take into account the "internal structure (and dynamics)" of, say, "a 'large' molecular ion".

These equations are intended as modifications of the Lorentz force equation in order to describe the effects of radiation to the lowest order. Even the lowest order is almost unobservable with present technology due to the extreme smallness of the effects. One can see the effects of radiation on particle orbits only in powerful accelerators. Basically, all that can be observed at present is that accelerated charged particles appear to lose energy at a rate consistent with the Larmor law. The Lorentz-Dirac equation, and all proposed modifications of it, are probably consistent with this observation, and otherwise experimentally indistinguishable, with present technology.

Any effects of internal structure will almost certainly be higher-order effects which are correspondingly less observable. It seems unreasonable to demand a theory to take into account higher order effects when even the lowest order effects are poorly understood and barely observable, if at all.

An analogy is a derivation of the basic gas laws assuming that the gas consists of spherical particles that collide elastically. This gives a sensible and very useful theory, even though everyone recognizes that internal structure of actual gas molecules may cause higher-order deviations.

In the early 1950's, Dirac spent a lot of time trying to construct a sensible classical electrodynamics because he thought that the logical problems of quantum electrodynamics might be artifacts of quantizing an *incorrect classical electrodynamics*. This hope has faded over the years due to lack

of progress and the difficulties of experimental verification. Nevertheless, it was a reasonable hope at the time, and even today, it would seem rash to dismiss it out of hand.

The problems which Rohrlich's proposed DD equation were intended to solve may not be of interest to the referee, but there does exist a community which still is interested in them. Indeed, I recall a recent paper by S. Sonego in *J. Math. Phys.* (JMP) establishing the conformal invariance of the right side of a curved-spacetime version of the Lorentz-Dirac equation. (Despite its appearance, this is not routine.) I was very interested when I saw this result in JMP because I had been working on it myself, for application to other problems.

This interesting and useful paper could not have been published under the standards of the second referee for JMP, since, according to him, analyses of equations of motion like the Lorentz-Dirac equation are not "relevant for physics". I hope that this referee's narrow view will not become the editorial policy of *J. Math. Phys.*.

If papers dealing with predicted phenomena beyond the possibility of observation with present technology were disallowed from physics journals, most theoretical journals, including JMP, would be only a small fraction of their present sizes. In particular, it would be virtually impossible to publish a paper on string theory.

Admittedly, some might consider this a good thing, and maybe it would be. But my point is that there seems a certain inconsistency of standards. If the standard of relevance applied to this paper by *J. Math. Phys.* and *Phys. Rev. D* were applied uniformly to all papers, very few would be published.

11.2 Reports of the referees for *Phys. Rev.* D (PRD)

11.2.1 Report of F. Rohrlich, identified referee

Referee's report: "From F. Rohrlich to the author, S. Parrott:

- Eq. (6) is not MY eq. but the Caldirola-Yaghjian eq., derived by A. D. Yaghjian (in his Appendix D) as referenced in my paper (Parrott's ref. 11)
- 2. The transition (6) to (7) prevents an analysis of (6) for the physically meaningful case: $m_2\tau_1 \ll m_1$ [in eq. (6)] except in the limit $m \gg 1$ [in the units of eq. (7)]. Physically, this means that the self-force is a small correction to the external force. Thus, while the mathematics of the paper may be correct, the results, based on eq. (7) are physically meaningless except when $m \gg 1$ "
- Author's comment: Regarding Objection 1, earlier versions of this paper called the DD equation "Rohrlich's equation" for reasons discussed in the

Introduction. I had sent a copy to Professor Rohrlich over a year previously, and had received no comment. This objection to the naming of the equation is the first I have received. In deference to his wishes, I changed the name to the uninspired but hopefully inoffensive "DD equation".

Objection 2 makes no sense to me. Rohrlich's original equation, which is equation (6) of my paper, contains three positive parameters named (in my notation) m_1, m_2 , and τ_1 . These three parameters are redundant and so I simplified equation (6) by choosing units and renaming quantities so as to replace m_1, m_2 , and τ_1 by a single parameter $m := m_1/(m_2\tau_1)$. The details of how this is done are spelled out in Appendix 2 (which was added in response to this report).

Note that $m_2\tau_1 \ll m_1$ is the same as $m \gg 1$. So Rohrlich's objection is essentially equivalent to:

The physically meaningful case is $m \gg 1$, and the paper's results are physically meaningless except when $m \gg 1$.

This may be tautologically true as a formal logical statement, but it has no relevance to the physical applicability of the paper's analysis.

Another way to see that the objection is invalid is to note that the paper's analysis holds for *all* values of the original parameters m_1, m_2 , and τ_1 . In any physically possible situation, m_1, m_2 , and τ_1 will have *some* values, and m will have some value. The analysis of the paper applies to these physically possible values because it applies to *all* values.

11.2.2 Report of Referee 2 for Phys. Rev. D

Referee's report "The paper contains a mathematical discussion of the asymptotic properties of the solutions of an equation of motion for charged particles related to a proposal of Rohrlich. It is shown that in 1 + 1 dimensions the existing theorems on equations allow to state that there are no runaway solutions, but that phenomena like either preacceleration or postacceleration are present.⁸ Due to the essentially mathematical character of the paper and since these types of equations are interesting for the foundations of physics but not for phenomenology (the Lorentz-Dirac

⁸Author's Comment: This gives the false impression that the results of the paper are consequences of "[pre-]existing theorems", i.e., not original. It is also an inaccurate summary of what the paper establishes.

The paper notes that solutions which are runaway in the past do exist, but in special cases can be eliminated by proper choice of generalized initial conditions. It notes the phenomenon of postacceleration for the first time in the literature. The theorem on nonexistence of future runaways only applies for eventually vanishing force and was not only not "existing" before the paper, but has a mathematically novel proof.

Some may consider these small inaccuracies, but a conscientious referee ought to get things like this right. There is no internal evidence in this report that the referee read the paper even to the depth necessary to correctly describe its results.

equation, notwithstanding its pathologies, is consistent with the conservation of energy-momentum and gives satisfying results at least till when [sic] classical theory breaks down due to pair production)." [The last sentence lacks both a subject and a verb—this is not an error in transcription.]

"Therefore the paper is not appropriate for Physical Review D. It should be sent to a more mathematically oriented journal."

11.2.3 First report of Referee 3 for Phys. Rev. D

Referee's report: Rohrlich makes clear beyond any doubt in the references [11]-[13] that any classical equation for the motion of charged particles can be physically reasonable and applicable only for a limited range of its parameters (mass, charge, 'radius' of particles, etc.). By the way, this is true for any physical equation. (This is one important difference between physics and mathematics!) Therefore I completely agree with Rohrlich's critique that 'results based on eq. (7) are physically meaningless except when $m \gg 1$ (and this means that m is bigger than any realistic particle mass by a factor of 1020 [sic: presumably the referee meant 10^{20}] or so!) I also agree with the critique of the second (anonymous) referee that the main body of the paper (Chaps. 2 and 3, and especially Chaps. 4 and 5) is pure mathematics that does (according to Rohrlich's remark) not solve any physically interesting problem.

Therefore I recommend rejection of this paper by Phys. Rev. D.

Author's comment: First of all, m is not a "particle mass" as the referee thought; it is just a parameter which I happened to name m. Indeed, m doesn't even have the dimensions of mass (it is dimensionless).

It is entirely irrelevant that m may be large in physically realistic situation. As pointed out in my comment on Rohrlich's objection, the paper's analysis applies to *all* values of m, so in particular it applies to all physically realistic values.

Given that the paper went to this new referee specifically to adjudicate the dispute over the equivalence of (6) and (7) (as described in the "History" subsection 10.1), it is shocking that he got this wrong. (In his next report he does reverse himself on this point.) Moreover, he appears to have simply ignored Appendix 2's careful explanation of this equivalence.

It was at this point that I took the unusual step of requesting an identified referee of mutually agreed competence (I suggested Barry Simon) to adjudicate the narrow issue of the equivalence of (6) and (7). Unfortunately, the Editor, D. Nordstrom, sent the paper back to this same referee.

I began to wonder if the Editor were really interested in obtaining an accurate referee's report. When a referee questions such an elementary equivalence even after a careful explanation (Appendix 2) is furnished, how can an editor be sure of his understanding of the rest of the paper,

and how can an editor trust his judgment? In fairness to the author, shouldn't the paper be sent to yet another referee, however tedious and annoying this might be for the editor?

11.2.4 Second report of Referee 3 for Phys. Rev. D

- Referee's report: "Three referees agree by now that this paper contains too little new and interesting physics (and too much simple but tedious mathematics) to justify publication in Phys. Rev. D.⁹ How many experts are necessary to convince the author of this fact? A fact which should also be evident from a comparison of this manuscript with an average article published in Phys. Rev. D. The topic of a classical equation of motion for charged particles does not belong to the most urgent questions in today's theoretical physics: All questions being of practical (let alone experimental) interest are solved by equations from the literature. And, due to the omission of quantum effects, there is not much interest in questions of principle. I have no doubt about the mathematical equivalence of equs. (6) and (7). (And I have read and confirmed the calculations in Appendix 2!) But the author should agree that equ. (7) is very unusual¹⁰ due to the (dimensionless) factor m which for all cases of physical interest is of the order of 10^{20} . And therefore the question arises (compare Abstract): For external forces of what strength, applied for what time, do preaccelerations or postaccelerations of what magnitude occur? And are these (physically unpleasant) effects relevant for any situation of practical physical interest? The author does nothing to address these questions.
- Author's comment: The referee raises one new, and valid, issue of whether the unphysical effects predicted by the DD equation can be observed with present technology. The paper doesn't address this issue because I don't know how to estimate the size of such effects. It is rare that a paper will solve all problems connected with a particular issue—one solves what one can.

My interest in Rohrlich's proposal was his claim to have produced an electrodynamical equation of motion which predicted no unphysical effects. Now that this claim has been shown to lack foundation, I am not motivated to further study the DD equation. It should be the responsibility of the *proponents* of this equation to show that any unphysical effects which it predicts must necessarily be too small to measure.

⁹Author's comment: One of these three referees was Professor Rohrlich. One assumes that he, at least, must regard as interesting the question of whether the equation of motion which he proposed has unphysical solutions. The referee's remark might be accurate if he replaced "three referees" by "two out of three". Some may think the inaccuracy a small thing, but it might also be regarded as betraying bias.

¹⁰Author's comment: I don't agree. I don't see anything unusual about it.

12 Appendix 5: Solving the DD equation backwards in time

The referees for *Physical Review* D (PRD) raised the following objections (and only these) to the present paper.

1. The version of the DD equation analyzed by the paper, (7), is not physically equivalent to the one proposed by Rohrlich, (6).

After many months and many pages of careful explanation, the only independent referee to affirm this objection finally withdrew it. So, one hopes that this issue may be regarded as settled.

- 2. The subject of the correct equation of motion for classical charged particles is of insufficient physical interest.
- 3. The paper is too mathematical.
- 4. The paper does not estimate the size of the postaccelerations predicted by the DD equation.

Objection 2 seems odd, given that PRD recently published Rohrlich's paper [13] on precisely this topic (equations of motion for classical charged particles), along with papers on the same topic by other authors. However, taking it at face value, it occurred to me that it certainly shouldn't apply to a "Comment" paper, e.g., "Comment on *Classical self-force* by F. Rohrlich". (If it did apply, that would amount to an editorial declaration that the subject of *Classical self-force* was of insufficient physical interest to merit publication, not long after PRD had published it!) Also, such a paper, (which is required to be short and therefore could not contain a full mathematical analysis) would obviate Objection 3.

I wrote to Editor D. Nordstrom asking if PRD would consider such a "Comment" paper, or if perhaps the editors had decided not to publish any form of the present work. After he didn't reply, I rewrote the paper as a Comment paper and submitted it on July 28, 2004. Its submission was acknowledged immediately, but by the end of December, I had heard nothing more. I wrote on December 28 enquiring about the status of the paper, and in particular if it had been sent to any referee. On January 28, 2005, I received a cryptic reply apologizing that the paper had been "misfiled". and stating that it had been "sent out for review". (This is being written on January 31.)

The presentation of the Comment paper is based on the fact that the solution manifold for the DD equation is *infinite-dimensional*, rather than of finite dimension as expected (cf. Appendix 1). Thus to base a physically reasonable theory on this equation, one needs some auxiliary condition to reduce the solution manifold to finite dimension.

That the DD equation's solution manifold is of infinite dimension is implicit in the present work and strongly suggested by the fact that the space of generalized initial conditions is obviously infinite-dimensional. However, no careful proof that the solution manifold is infinite-dimensional was included above because it was only used for motivation in peripheral discussions.

The "Comment" paper contains a broad mathematical outline, but no substantial proofs, for which the reader is referred to the present work. After submitting it, I realized that the present work didn't give a real proof that the solution manifold of the DD equation is infinite-dimensional. This appendix remedies this by furnishing a proof.

Although the Comment paper uses Rohrlich's original form (6) of the DD equation, for notational simplicity, we use here the normalized DD equation (7), so that the time units are normalized to make the delay τ_1 equal to 1:

$$m\frac{du^{i}}{d\tau} = f^{i}(\tau) + [u^{i}(\tau-1) - u^{\alpha}(\tau-1)u_{\alpha}(\tau)u^{i}(\tau)] \quad , \tag{7}$$

Recall that a generalized initial condition is a specification of u on some interval $\tau_0 - 1 \leq \tau \leq \tau_0$ of length 1. (For a general delay τ_1 , the interval of specification would be an interval of length τ_1 .) Section 3 noted that to every such generalized initial condition corresponds a unique solution defined on $[\tau_0, \infty)$. In other words, the DD equation can be solved *forward* in time starting with any generalized initial condition.

We are now going to show how to solve the DD equation *backward* in time to obtain a unique solution on $(-\infty, \tau_0)$ for each generalized initial condition. In Section 3, this was done in a simple, *ad hoc* way for the DD1 equation, which was all that was needed for the treatment given. But the Comment paper doesn't mention the DD1 equation, so now we need to do it for the full DD equation (7).

Combining the forward solution with the backward solution gives a unique solution on $(-\infty, \infty)$ corresponding to each generalized initial condition on $(\tau_0 - 1, \tau_0)$. We define the solution manifold to be the set of all solutions defined on $(-\infty, \infty)$.

Then it becomes obvious that the solution manifold is parametrized by the set of all generalized initial conditions on any given interval $(\tau_0 - 1, \tau_0)$ of length 1. The technical issue is whether the assignment of a solution to each generalized initial condition is one-to-one (hence bijective). This is obvious if "solutions" are required to be defined on $(-\infty, \infty)$ because then each generalized initial condition is the restriction of a solution. (It's not obvious if "solutions" are only required to be defined on (τ_0, ∞) , though it can be proved.)

To see how to solve the DD equation backward in time starting from a generalized initial condition, attempt to solve in (7) for $u(\tau-1)$ in terms of $u(\tau)$, $du/d\tau$, and $f(\tau)$. This is impossible because the bracketed quantity is only the projection of $u(\tau-1)$ on $u(\tau)$, but the equation does uniquely determine this projection. Hence it uniquely determines $u(\tau-1)$ up to an additive term $\beta u(\tau)$, β a scalar:

$$u(\tau - 1) = \beta u(\tau) + [m\frac{du^i}{d\tau} - f^i(\tau)] = \beta u(\tau) + q(\tau) \quad ,$$

where for brevity we introduce the name

$$q(\tau) := [m\frac{du^i}{d\tau} - f^i(\tau)]$$

for the bracketed quantity.

Routine calculation reveals that the scalar β is uniquely determined by the necessary condition that the four-velocity $u(\tau - 1)$ must be a future-pointing unit vector. The condition that $u(\tau - 1)$ have unit norm gives a quadratic equation for β , and the condition that $u(\tau - 1)$ be future-pointing singles out a unique solution to this quadratic equation. The result of this calculation is:

$$u(\tau - 1) = \left[-u^{\alpha}(\tau)q_{\alpha}(\tau) + \sqrt{1 + (u^{\alpha}(\tau)q_{\alpha}(\tau))^{2} - (q^{\alpha}(\tau)q_{\alpha}(\tau))^{2}}\right]u(\tau) + q(\tau).$$

Thus given the four-force f, the values of $u(\tau)$ on an interval $[\tau_0, \tau_0 + 1]$, uniquely determine the values of $u(\tau - 1)$ on this interval, Equivalently stated, the values of $u(\tau)$ on $[\tau_0, \tau_0 + 1]$ uniquely determine the values of $u(\tau)$ on $[\tau_0 - 1, \tau_0]$, and, by iteration, the values of $u(\tau)$ on $(\infty, \tau_0]$

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