COMMENT

Proof gap in "Sufficient conditions for uniqueness of the Weak Value" by J. Dressel and A. N. Jordan

2575 Bowers Rd. Gardnerville, NV 89410, USA
Department of Mathematics (retired), University of Massachusetts at Boston (for identification, not mail)
http://www.math.umb.edu/∼sp

E-mail: S_Parrotttoast2.net

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Abstract. The commented article attempts to prove a "General theorem" giving sufficient conditions under which a previously introduced "general conditioned average" "converges uniquely to the quantum weak value in the minimal disturbance limit." The "general conditioned average" is obtained from a positive operator valued measure (POVM) $\{\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g)\}_{j=1}^{n}$ depending on a small "weakness" parameter g. We point out that unstated assumptions in the presentation of the "sufficient conditions" make them appear much more general than they actually are. Indeed, the stated "sufficient conditions" strengthened by these unstated assumptions seem very close to an assumption that the POVM be a *linear* polynomial (i.e., of first order in g). Moreover, there appears to be a critical error or gap in the attempted proof, even assuming a linear POVM. A counterexample to the *proof* of the "General theorem" (though not to its conclusion) is given. Nevertheless, I conjecture that the conclusion is actually true for *linear* POVM's whose contextual values are chosen by the commented article's "pseudoinverse prescription".

1. Introduction

Notation will be the same as in the article under review [1], called DJ below. To compress this Comment to a traditional length, we must assume that the reader is already familiar with DJ. Its main purpose seems to be to justify a statement of [2] (called DAJ below) that a "general conditioned average" introduced in DAJ "converges uniquely to the quantum weak value in the minimal disturbance limit". DJ formulates "sufficient conditions" as hypotheses for a "General theorem" (GT) with this statement as its conclusion.

For simplicity, we shall only consider the special case of DAJ and DJ's "minimal disturbance" condition, for which all measurement operators $\{\hat{\boldsymbol{M}}_{\boldsymbol{j}}\}$ are positive. (All statements which we make will also hold for DJ's slightly more general definition.) The associated positive operator valued measure (POVM) is $\{\hat{\boldsymbol{E}}_{\boldsymbol{j}}\}$ with $\hat{\boldsymbol{E}}_{\boldsymbol{j}} := \hat{\boldsymbol{M}}_j^{\dagger}\hat{\boldsymbol{M}}_{\boldsymbol{j}}$. The measurement operators $\hat{\boldsymbol{M}}_{\boldsymbol{j}} = \hat{\boldsymbol{M}}_{\boldsymbol{j}}(g)$ depend on a small "weakness" parameter g which quantifies the degree to which the measurement affects the system being measured. Our "minimal disturbance limit" will refer to the so-called "weak limit" $g \to 0$ for positive measurement operators.

DAJ claims that under these assumptions, its "general conditioned average" (corresponding to what is more usually called a "weak" measurement followed by a postselection) is given by the traditional "quantum weak value" (the real part of DJ's (1.1)) in the weak limit $g \to 0$. Counterexamples to this claim were given in [3], examples which DJ attempts to refute by reinterpreting the hypotheses of its "General theorem"(GT).

The mathematics of these counterexamples is undisputed; the only issue is whether they satisfy the hypotheses of DAJ or DJ. DJ correctly notes that the first counterexample using 2×2 measurement matrices does not satisfy what they call the "pseudoinverse prescription", but DAJ does not clearly state this prescription as a hypothesis. The second counterexample using 3×3 matrices does satisfy the pseudoinverse prescription, so the following will deal exclusively with this counterexample.

Contrary to claims of DJ, this counterexample *is* valid when the hypotheses of the "General theorem" (GT) are interpreted *as written*, according to standard usage of logical language. However, DJ's analysis of the counterexamples requires a great strengthening of one of these hypotheses, a strengthening not noted in DJ. We shall see that when so strengthened, the hypotheses of the GT seem very close to the assumption that the POVM must be a *linear* polynomial in the weak measurement parameter g, i.e.,

$$\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g) = \hat{\boldsymbol{E}}_j^{(0)} + g\hat{\boldsymbol{E}}_j^{(1)} \text{where } \hat{\boldsymbol{E}}_j^{0} \text{ and } \hat{\boldsymbol{E}}^{(1)} \text{ are constant operators.} \tag{1}$$

The analysis leading to the above conclusions will be straightforward and simple. DJ's attempted proof of the GT is densely written, and our analysis of it must be correspondingly technical. Although probably few readers will be sufficiently familiar with the proof to convince themselves either of its truth or of the claim that there is a major error, I hope that the analysis may motivate anyone tempted to employ the "General theorem" in other work to first carefully scrutinize the proof.

2. Unstated hypotheses for the "General theorem"

The hypotheses of the "General theorem"(GT) which will concern us are:

"(iii) The equality $\hat{\boldsymbol{A}} = \sum_j \alpha_j(g)\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g)$ must be satisfied, where the contextual values $\alpha_j(g)$ are selected according to the pseudo-inverse prescription.

(iv) The minimum nonzero order in g for all $\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g)$ is g^n such that (iii) is satisfied."

"Minimum nonzero order" is not a standard mathematical phrase, but I take its occurrence in (iv) to mean that

$$\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g) = \hat{\boldsymbol{E}}_j^{(0)} + g^n \sum_{k=0}^{\infty} \hat{\boldsymbol{E}}_j^{(k+n)} g^k \tag{2}$$

with $\hat{\boldsymbol{E}}^{(k+n)}$ constant operators and $\hat{\boldsymbol{E}}_j^{(n)} \neq 0$. Then the logical content of (iv) is that all $\hat{\boldsymbol{E}}_{\boldsymbol{j}}$ have the *same* minimum nonzero order, which is to be denoted n.‡ This is a

‡ The restrictive phrase "such that (iii) is satisfied" is logically redundant, since (iii) has *already* been assumed. If the authors mean that some alteration of (iii) is to be assumed, such as (iii) with the $\hat{\boldsymbol{E}}_{\boldsymbol{j}}$ replaced by their truncations to order n or (iii) with the original contextual values previously denoted

strange and quite restrictive assumption, but it will not be our main concern. From here on, we simplify notation by only considering the case $n = 1$.

When the hypotheses of the GT are given the standard logical interpretation just described, the counterexample using 3×3 matrices which DJ attempts to refute is indeed a counterexample to the GT. However, DJ's attempt to refute the counterexample appears to assume something like the following,

> Denote by $\hat{\boldsymbol{E}}'_j(g)$ the truncation of the series (2) to order n, namely,
>
> $$\hat{\boldsymbol{E}}'_j(g) := \hat{\boldsymbol{E}}_j^{(0)} + \hat{\boldsymbol{E}}_j^{(n)} g^n \quad . \tag{3}$$
>
> Then (iv) assumes (iii) with the $\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g)$ in (iii) replaced by $\hat{\boldsymbol{E}}'_j(g)$, but with the contextual values $\alpha_j(g)$ unchanged (i.e., the contextual values for the truncated POVM $\{\hat{\boldsymbol{E}}'_j(g)\}$ are the same as for the original POVM $\{\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g)\}$). More explicitly, it assumes that
>
> $$\sum_j \alpha_j(g) \hat{\boldsymbol{E}}'_j(g) = \hat{\boldsymbol{A}}. \tag{4}$$

DJ's objection to the counterexample, given after its equation (7.4), is that it does not satisfy (4). DJ does not explicitly say that the contextual values for the truncated POVM are the same as for the original POVM, but that seems implied by the fact that it uses the same symbols, $\alpha_j(g)$ for both. Also, the details of DJ's attempted proof support that interpretation.

Next recall that the contextual values $\vec{\alpha}(g) = (\alpha_1(g), \ldots, \alpha_n(g))$ are assumed to satisfy the "pseudoinverse prescription"

$$\vec{\alpha} = F^+ \vec{a}, \tag{5}$$

where $\vec{a}$ is a list of eigenvalues for the system observable $\hat{\boldsymbol{A}}$, and F^+ is the Moore-Penrose pseudoinverse for the matrix

$$F = F(g) := [\vec{E}_1(g), \ldots, \vec{E}_n(g)]. \tag{6}$$

Here the column vector $\vec{E}_j(g)$ is the list of eigenvalues for $\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g)$, and F is the matrix composed of those columns. Note that F is g-dependent, but we write $F = F(g)$ only when necessary to emphasize this point, to avoid possible confusion with the result of applying the matrix F to a vector. If contextual values exist (in general, they don't), they are *uniquely determined* by the "pseudoinverse prescription" (5).

Since the contextual values for the truncated POVM $\{\hat{\boldsymbol{E}}'_j(g)\}$ are assumed the same as those for the original, we also have

$$\vec{\alpha} = F'^+ \vec{a} \tag{7}$$

with

$$F' := [\vec{E}'_1(g), \ldots, \vec{E}'_n(g)], \tag{8}$$

where the $\vec{E}'_j(g)$ are the column vectors of eigenvalues for $\hat{\boldsymbol{E}}'_j(g)$. Equation (7) also *uniquely determines* the contextual values $\alpha_j(g)$, so it would be surprising if *both* (5) and (7) would hold except in the trivial case in which $\hat{\boldsymbol{E}}_{\boldsymbol{j}} = \hat{\boldsymbol{E}}'_j$ for all j. In that case, we can make $\{\hat{\boldsymbol{E}}_{\boldsymbol{j}}\}$ linear (i.e., of form (1)) by replacing the parameter g by a new

$\alpha_j(g)$ replaced by others or some combination of these, then standard logical language requires that this be explicitly stated. I have considered several alternative interpretations of (iv), but all have led to inconsistencies with other parts of DJ. In the absence of requested clarification from the authors, I selected the one which seems most nearly consistent with the rest of DJ.

parameter $h := g^n$, so for brevity we shall refer to this as the "linear case". Indeed, I do not know of any example of a nonlinear POVM for which both (5) and (7) can hold. The hypothesis that both do hold seems very close to a hypothesis that the original POVM be linear.

3. Error or gap in proof

Readers thinking of building on the work of DAJ and DJ may need to convince themselves of the validity of its "General theorem". Since its attempted proof is densely written, it may be of help to pinpoint what I think is a critical error (or at least a serious gap), even under the strong hypothesis that the POVM is linear.

This is equivalent to the assumption that the matrix $F = F(g)$ determining the contextual values $\vec{\alpha}$ is first order in g, in which case the minimum nonzero order of F which the proof calls n is $n = 1$. To expose the gap, we use these assumptions to rewrite the questionable part of the proof in a simplified form. It applies to a matrix F with singular value decomposition $F = U\Sigma V^T$, where Σ is a diagonal matrix and "U and V are orthogonal matrices". All of these matrices depend on the weak limit parameter g.

The contextual values $\vec{\alpha}$ (which the proof renames $\vec{\alpha}_0$) are determined by the pseudoinverse prescription $\vec{\alpha} = \vec{\alpha}_0 = F^+\vec{a}$, where $\vec{a}$ is the vector of eigenvalues of A. Here F^+ is the Moore-Penrose pseudoinverse of F, given by $F^+ = V\Sigma^+U^T$, where Σ^+ is the diagonal matrix obtained from Σ by inverting all its nonzero elements.

In reading the following, please keep in mind that if correct, it should apply to *any* matrix function $F = F(g)$ which is first order in g, i.e., $F(g) = P + gQ$ with P and Q constant matrices. Although an $F = F(g)$ derived from a POVM has a special form given in part by DJ's preceding equation (5.9), *nothing in the following proof fragment uses this special form.*

The proof mentions "relevant" singular values, but for brevity I have omitted the definition of "relevant" (which does not involve the special form of F) because for the simple counterexample to be given, it will be apparent that every singular value will be "relevant" for some $\vec{a}$ (because F will be invertible). The simplified proof fragment is:

> Since the orthogonal matrices U and V have nonzero orthogonal limits $\lim_{g\to 0} U = U_0$ and $\lim_{g\to 0} V = V_0$, such that $U_0^T U_0 = V_0 V_0^T = 1$, and since $\vec{a}$ is g-independent, then the only poles in the solution $\vec{\alpha}_0 = F^+\vec{a} = V\Sigma^+U^T\vec{a}$ must come from the inverses of the relevant singular values in Σ^+.
> Therefore, to have a pole of order higher than $1/g$, there must be at least one relevant singular value with a leading order greater than g^1.
>
> [This much seems all right, but I cannot follow the next and last paragraph.]
>
> However, if that were the case then the expansion of F to order g^1 would have a relevant singular value of zero and therefore could not satisfy (5.12), contradicting the assumption (iv) about the minimum nonzero order of the POVM. Therefore, the pseudoinverse solution $\vec{\alpha}_0 = F^+\vec{a}$ can have no pole with order higher than $O(1/g)$ and the theorem is proved.

If correct, the above proof fragment would imply that no linear matrix function $F(g) = P + gQ$, with P and Q constant matrices, could have a singular value with a leading order greater than g^1. However, it is easy to construct counterexamples such as

$$F := \begin{bmatrix} 1+g & 1 \\ -1 & -1+g \end{bmatrix}, \tag{9}$$

which has a singular value $[g^2+2-2\sqrt{g^2+1}]^{1/2} = g^2/2+O(g^4)$. (The other singular value is $[g^2+2+2\sqrt{g^2+1}]^{1/2} = 2+g^2/2+O(g^4)$.)

Without performing the somewhat messy calculation of the singular values, one can see directly from Cramer's rule that since $\det F(g) = g^2$, $F(g)^{-1} \sim g^{-2}$ which would make the contextual values $\vec{\alpha} = F^{-1}\vec{a}$ asymptotic to g^{-2}. The essence of the full proof of the GT is to show (continuing to assume $n = 1$ for simplicity) that the contextual values are $O(1/g)$, which implies that the "numerator correction" of DJ's (5.7) vanishes in the limit $g \to 0$.

Let us try to follow in detail the last paragraph of the proof in the context of the counterexample. First note that DJ's equation (5.12) is just the contextual value equation $F(\vec{\alpha}) = \vec{a}$ written in components.

Applied to the F of (9), the last paragraph asserts that if F has a singular value of order greater than g^1 (which it does), then "the expansion of F to order g^1 would have a relevant singular value of zero ...". However, this is wrong because the expansion of F to order g^1 is F itself, and for $g \neq 0$, all singular values are positive.

I suspect that the last paragraph may be based on an erroneous implicit assumption that truncating a Σ corresponding to $F(g)$ will produce the Σ for the truncated F, i.e., that truncation commutes with taking of singular values. If this were true, then the statement just questioned would be true.

To make the above more explicit, write $\Sigma = \Sigma(F)$ to denote the dependence of the matrix Σ of singular values on F, and write $\tau(F)$ for the linear truncation of F. I can begin to make sense of the last paragraph only by assuming that

$$\Sigma(\tau(F)) = \tau(\Sigma(F));$$

which says that the singular values for the truncated F are the truncations of the singular values for F. The counterexample shows that this is false for its F which satisfies $\tau(F) = F$:

$$\Sigma(\tau(F)) = \Sigma(F) = \begin{bmatrix} 2+g^2/2+O(g^4) & 0 \\ 0 & g^2/2+O(g^4) \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \tau(\Sigma(F)).$$

Recall that (4) was my best guess at the intended expansion of DJ's (iv) from its logical meaning. (A direct request to the authors to confirm or correct this was ignored.) My next best guess would be that the $\alpha_j(g)$ in (4) might represent contextual values for the truncated POVM $\{\hat{\boldsymbol{E}}_j'\}$ that would not necessarily be contextual values for the original POVM $\{\hat{\boldsymbol{E}}_{\boldsymbol{j}}\}$. However, the above objection to the proof would still apply.

Whatever the intended meaning of DJ's (iv), it presumably imposes some condition on the linear truncation (still taking $n = 1$ for simplicity) of the original POVM $\{\hat{\boldsymbol{E}}_{\boldsymbol{j}}(g)\}$, which seems an unreasonable hypothesis for a "General theorem". Suppose one has an experiment whose detector corresponds to a certain nonlinear POVM. Why should it matter if the linear truncation of this POVM has some property? A theorem whose hypotheses eliminate even simple cases such as the second counterexample of [3] with a quadratic POVM can hardly be considered "General". The strong impression given by both DAJ and DJ is that the traditional weak value (the real part of DJ's (1.1)) is essentially inevitable when the measurement operators are positive. A main point of both [3] and the present Comment is to dispel any such false impression.

It should be emphasized that (9) is only a counterexample to DJ's attempted *proof*, not a counterexample to the *conclusion* of the GT under the assumption that the POVM is linear, i.e., of the form (1). For a counterexample to the conclusion, one would need an F which is derived from a POVM.

Actually, I conjecture that the conclusion that the "general conditioned average" is given by DJ's (1.2) (i.e., the traditional weak value generalized to mixed states) is *true* for *linear* POVM's under the pseudoinverse prescription. If so, its proof will surely have to use in some essential way the special form of an F which comes from a POVM (e.g., all rows sum to 1).

I have sketched such a proof but have not written it in detail, so I make no claims. I will be happy to share the ideas of the proof with any qualified person who might be interested in expanding on them. They are not difficult, but annoyingly detailed. If I decide not to write them up in journal-ready detail myself, I may put a sketch of a proof on my website, www.math.umb.edu/∼sp.

A main aim of this comment is to focus attention on the case of linear POVM's. If the conjecture is true, it might help to explain why (to my knowledge) actual experiments have only observed the traditional weak value (the real part of DJ's (1.1), $\langle\psi_f|\hat{\boldsymbol{A}}|\psi_i\rangle/\langle\psi_f|\psi_i\rangle$), despite the fact that arbitrary weak values can be obtained from different measurement procedures, as stressed by DJ. Since these experiments are difficult and have only recently been performed, perhaps they correspond to the simplest POVM's, i.e., *linear* POVM's arising from positive measurement operators.

References

[1] Dressel J and Jordan A N 2012 Sufficient conditions for uniqueness of the Weak Value, *J. Phys. A: Math. Theor.* **45** 015304 1-14, called DJ in the text.

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[3] Parrott S Counterexample to 'Sufficient conditions for uniqueness of the Weak Value' by J. Dressel and A. N. Jordan, *Preprint* arXiv:1106.1871