COMMENT

Comment on "Sufficient conditions for uniqueness of the Weak Value" by J. Dressel and A. N. Jordan: Minimizing detector variance

2575 Bowers Rd. Gardnerville, NV 89410, USA
Department of Mathematics (retired), University of Massachusetts at Boston (for identification, not mail)
http://www.math.umb.edu/∼sp

E-mail: S_Parrotttoast2.net

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Abstract. The commented paper proposes choosing its "contextual values" according to a "pseudoinverse prescription" to attempt to minimize the "detector variance". This note proposes an alternative method that seems better.

The commented paper [1], called DJ below, includes a section attempting to justify choosing contextual values according to its "pseudoinverse prescription". This note proposes a simple method which seems better. We assume the reader is familiar with DJ and use its notation without comment.

DJ considers a collection of measurement operators $\{\hat{\boldsymbol{M}}_j\}_{j=1}^n$ associated with a "detector". The detector is designed to measure the expectation $\langle A\rangle$ of a "system observable" $\hat{\boldsymbol{A}}$.

Let ρ denote the (mixed) state of the system and $P(j) = \mathrm{Tr}[\rho\hat{\boldsymbol{M}}_j^\dagger\hat{\boldsymbol{M}}_j]$ the probability that the jth measurement outcome is observed. (Though $P(j)$ depends on the state ρ, this dependence is not included in DJ's notation.) DJ considers the situation in which it is possible to choose "contextual values" α_j such that for all states ρ,

$$\langle A\rangle = \sum_j \alpha_j P(j). \tag{1}$$

This gives a way to measure $\langle A\rangle$ by approximating $P(j)$ by the observed frequency of the jth outcome.

The contextual values $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ are solutions to a "contextual value" equation

$$F\vec{\alpha} = \vec{a} \tag{2}$$

where $\vec{a} = (a_1, a_2, \ldots, a_m)$ is a vector composed of the eigenvalues of a "system observable" $\hat{\boldsymbol{A}}$, and F is a certain $m \times n$ matrix. DJ's "pseudoinverse prescription" requires that when the choice of contextual values is not unique, $\vec{\alpha}$ should be chosen as

$$\vec{\alpha} = F^+\vec{a}, \tag{3}$$

where F^+ denotes the Moore-Penrose pseudoinverse of F. DJ's justification for the pseudoinverse prescription is that is minimizes a particular upper bound‡ $||\vec{\alpha}||^2 := \sum_j \alpha_j^2$ for the "detector variance" σ^2:

$$||\vec{\alpha}^2|| \geq \sigma^2 = \sum_j \alpha_j^2 P(j) - \langle A \rangle^2. \tag{4}$$

Of course, minimizing an *upper bound* for a quantity like detector variance does not imply that the quantity itself is minimized. Nevertheless, DJ concludes without proof or justification:

> "Therefore, by minimizing this upper bound the pseudoinverse will choose a solution that provides rapid statistical convergence for observable measurements on the system given no prior knowledge of the system state."

For experiments to which the ideas of DJ might be applied, one *does* have some (though not necessarily complete) prior knowledge of the system state. It is implicit in DJ's formulation that one has an apparatus that can generate many copies of a particular state; otherwise it would make no sense to speak of a "detector variance". If the state is finite-dimensional (which is implicit in DJ's mathematics), with sufficient copies one can determine to arbitrary accuracy its components with respect to any given basis by quantum process tomography, i.e., one can determine the state itself. Even for states in infinite dimensions, one can determine experimentally to any desired accuracy the *probabilities* $P(j)$ of the measurement outcomes j. Indeed, these probabilities are necessary to implement DJ's scheme to approximate $\langle A \rangle$ via (1). We shall observe that knowledge of the probabilities $P(j)$ is all one needs to choose contextual values to minimize the detector variance itself (not just an upper bound to the detector variance like $||\vec{\alpha}||^2$).

As DJ notes, minimizing σ^2 assuming that the contextual values $\vec{\alpha}$ are chosen to satisfy the contextual value equation $F\vec{\alpha} = \vec{a}$, is the same as minimizing the second moment which we denote by δ^2:

$$\delta^2 := \sum_j P(j)\alpha_j^2. \tag{5}$$

Let $\vec{\alpha}^{(0)}$ denote a particular solution of $F(\vec{\alpha}) = \vec{a}$. Then the general solution of $F(\vec{\alpha}) = \vec{a}$ is $\vec{\alpha} = \vec{\alpha}^{(0)} + \vec{\eta}$ with $\vec{\eta}$ in the nullspace Null(F) of F, and

$$\delta^2 := \sum_j P(j)\alpha_j^2 = \sum_j P(j)(\alpha_j^{(0)})^2 + 2\sum_j P(j)\alpha_j^{(0)}\eta_j + \sum_j p_j \eta_j^2. \tag{6}$$

For small $\vec{\eta}$, a nonvanishing linear second term will dominate the quadratic third term,§ and we see that if $\vec{\alpha}^{(0)}$ is to minimize σ^2, then the vector

$$(P(1)\alpha_1^{(0)}, \ldots, P(n)\alpha_N^{(0)}) \tag{7}$$

must be orthogonal to Null(F). Thus to minimize the detector variance, we should choose the contextual values to satisfy this condition.

This will rarely result in the pseudoinverse solution because the pseudoinverse solution $\vec{\alpha}_{\mathrm{PI}}$ is abstractly defined by the different condition that $\vec{\alpha}_{\mathrm{PI}}$ be orthogonal to Null(F). This fact is discussed in the present context but not proved in the Appendix to [4]. A formal statement and proof can be found in [3], p. 9, Theorem 1.1.1.

‡ A sharper upper bound for σ^2 is known [2], but it seems unlikely that the pseudoinverse prescription would minimize it.

§ More precisely, if for some η the linear term does not vanish, then replacing η by $x\eta$, with x real, gives a quadratic function in x with nonvanishing linear term, which cannot have a minimum at $x = 0$.

[1] Dressel J and Jordan A N 2012 Sufficient conditions for uniqueness of the Weak Value, *J. Phys.* A: *Math. Theor.* **45** 015304 1-14, called DJ in the text.

[2] Parrott S Counterexample to 'Sufficient conditions for uniqueness of the Weak Value' by J. Dressel and A. N. Jordan, *Preprint* arXiv:1106.1871

[3] Campbell S L and Meyer C D 2008 *Generalized Inverses of Linear Transformations*, Society for Industrial and Applied Mathematics

[4] Parrott S 'Contextual weak values' of quantum measurements with positive measurement operators are not limited to the traditional weak value *Eprint* arXiv:1102.4407