Calculational error invalidating claimed sufficient conditions to obtain the traditional weak value in Dressel and Jordan's "Contextual-value approach to the generalized measurement of observables"

Stephen Parrott
2575 Bowers Rd. Gardnerville, NV 89410, USA
Department of Mathematics (retired)
University of Massachusetts at Boston (for identification, not mail)
email: S_Parrott toast2.net
http://www.math.umb.edu/∼sp

25 March, 2013

The article of the title [1] presents a theorem which gives sufficient conditions for a "general conditioned average" introduced in [3] to yield the traditional weak value in the limit of arbitrarily weak measurements. This Comment notes a calculational error in a key lemma which invalidates the proof of the theorem.

PACS numbers: 03.65.Aa, 03.65.Ta

The article [1] presents a theorem which gives sufficient conditions for a "general conditioned average" introduced in [3] to yield the traditional weak value in the limit of arbitrarily weak measurements. The "general conditioned average" corresponds to what would more usually be called a (preselected and) postselected weak measurement. Since there is only one theorem in that article, we will simply call it "Theorem". Essentially the same result with the same attempted proof appears in different notation in [2]. This Comment notes a calculational error in a key lemma which invalidates the proof of the Theorem.

The statement of the Theorem is too complicated to repeat here in full, so we state only enough of it to explain the error. It concerns a positive operator valued measure (POVM) denoted $\{E_y(\epsilon)\}$, with the parameter y in a finite set of integers $1, \ldots, N$. The parameter $\epsilon > 0$ describes the "weakness" of the measurement; smaller ϵ correspond to weaker measurements.

To use the POVM to obtain the average value of an observable F_X (readers can ignore the subscript X, which is an artifact of the paper's unusual notation) one seeks so-called "contextual values" which the paper calls $f_Y(\epsilon; y)$ (again, the subscript Y can be ignored) such that

$$F_X = \sum_y f_Y(\epsilon; y) E_y(\epsilon) \quad \text{for all } \epsilon > 0. \tag{1}$$

For the purposes of this exposition, we shall assume POVM's of the so-called "linear" form, which means that

$$E_y(\epsilon) = E_y^{(0)} + \epsilon E_y^{(1)} \tag{2}$$

with $E^{(0)}, E^{(1)}$ constant (i.e., ϵ-independent) operators, though the Theorem purports to cover more general cases.

The essence of the Theorem's proof is to write the general conditioned average as a sum of two terms, one of which converges to the traditional weak value in the limit $\epsilon \to 0$, and the other of the form

$$\sum_y f_Y(y; \epsilon) \mathrm{Tr}[H_y(\epsilon)],$$

where the $H_y(\epsilon)$ are certain operators depending on the operators of the POVM and given by a certain formula. From the form of the formula, it is obvious that $H_y(\epsilon) = O(\epsilon^2)$, so the Theorem's conclusion will follow if it can be shown that the contextual values are $O(1/\epsilon)$.

The paper's Lemma 1 concludes that the contextual values are indeed $O(1/\epsilon)$ for linear POVM's, though it is stated slightly differently as a statement about the singular values of a matrix $\mathcal{S}'$ of the "linear" form $\mathcal{S}' = \mathcal{P} + \epsilon\mathcal{S}$ with $\mathcal{P}, \mathcal{S}$ constant matrices. The matrices $\mathcal{P}$ and $\mathcal{S}$ are not arbitrary, but are derived in a certain way from the POVM. The corresponding restrictions on $\mathcal{P}$ and $\mathcal{S}$ are not explicitly stated as hypotheses for Lemma 1, but appear earlier in the discussion.

Next we restate Lemma 1 including these restrictions explictly. The statement is identical to the statement of the same lemma in [5], except for replacement of that paper's notation with the present notation. That paper is a Corrigendum to [4], which contains essentially the same erroneous proof of the Theorem as the paper under discussion.

Restated Lemma 1. *The singular values of the $M \times N$ dimensional matrix $\mathcal{S}' = \mathcal{P} + \epsilon^n \mathcal{S}_n$ with $M \leq N$ have maximum leading order of ϵ^n, where $\mathcal{P} = [p_1\vec{1} \ldots p_N\vec{1}]$ and $\mathcal{S}_n = [\vec{E}_1 \ldots \vec{E}_N]$ such that $\sum_j p_j = 1$ and $\sum_j \vec{E}_j = \vec{0}$.*

Notes:

1. $\vec{1}$ stands for the column vector all of whose elements are 1, and $[\vec{E}_1 \ldots \vec{E}_N]$ for the matrix whose columns are the column vectors $\vec{E}_j$.
2. The parameter n plays no essential role and for simplicity can be set equal to 1.

The first few lines of the proof contain:

> "*Proof.* The singular values of $\mathcal{S}'$ are $\sigma_k = \sqrt{\lambda_k}$, where λ_k are M eigenvalues of $\mathcal{H} = \mathcal{S}^T\mathcal{S} \dots$"

The authors must mean $\mathcal{H} = (\mathcal{S}')^T\mathcal{S}'$ in place of $\mathcal{H} = \mathcal{S}^T\mathcal{S}$. The former is what makes mathematical sense, and is how the proof is worded in [5]. Continuing:

> "$\dots$ Since $\mathcal{P}^T\mathcal{S}_n = 0$, this matrix $[\mathcal{H}]$ has the simple form $\mathcal{H} = \mathcal{P}^T\mathcal{P} + \epsilon^{2n}\mathcal{S}_n^T\mathcal{S}_n$, $\dots$"

An example of such $\mathcal{P}$ and $\mathcal{S}_n$ is:

$$\mathcal{P} := \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \quad \mathcal{S}_n := \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \quad .$$

Then

$$\mathcal{P}^T\mathcal{S}_n = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} = \mathcal{S}_n \neq 0.$$

The attempted proof of the Lemma does not actually require that $\mathcal{P}^T\mathcal{S}_n = 0$, but only its consequence that $\mathcal{P}^T\mathcal{S}_n + \mathcal{S}_n^T\mathcal{P} = 0$. However, that is false, too. Since the rest of the proof of the Lemma rests heavily on the latter claimed equality, (equivalent to $\mathcal{H} = \mathcal{P}^T\mathcal{P} + \epsilon^{2n}\mathcal{S}_n^T\mathcal{S}_n$), it must be considered invalid. There appear to be several more subtle errors in the proofs of the lemmas and theorem, but there seems little point to discuss them until the simple error discussed above is corrected or bypassed. See [6] for further discussion.

The above assumed for simplicity that the POVM was of the "linear" form (2) because if the Theorem's proof is incorrect for this special case, then it is also incorrect under its more general hypotheses. Of course, that the Theorem's *proof* is incorrect does not imply that its *conclusion* is false.

In seeking a valid theorem of this type, it is natural to first consider the linear case. Under this hypothesis, [6] conjectures that the theorem is *true*.

Lemma 1 assumes that $M \leq N$. The attempted proof of the Theorem states that its hypothesis 4 implies that $M \leq N$, but this does not strictly follow from the logically ambiguous statement of 4, though it may be what the authors intended. The issue is whether hypothesis 4 is supposed to imply that contextual values exist for *all* system observables F_X, or only for a particular system observable which is fixed throughout the proof.

The most important case under the "linear" assumption is probably the case $M = N$ under the additional assumption that contextual values exist for all system observables F_X, and [6] proves Lemma 1 and the Theorem's conclusion for this case.

References

[1] J. Dressel and A. N. Jordan, "Contextual value approach to the generalized measurement of observables", Phys. Rev. A **85**, 022123 (2012)

[2] Dressel J and Jordan A N 2012 Sufficient conditions for uniqueness of the Weak Value, J. Phys. A: Math. Theor. **45** 015304 1-14

[3] Dressel J, Agarwal S, and Jordan A N Contextual Values of Observables in Quantum Measurements, Phys. Rev. Lett. **104** 240401 (2010)

[4] J. Dressel and A. N. Jordan, "Sufficient conditions for uniqueness of the weak value", J. Phys. A: Math. Theor. **45** (2012) 015304

[5] J. Dressel and A. N. Jordan, "Corrigendum: Sufficient conditions for uniqueness of the weak value", J. Phys. A: Math. Theor. **46** (2013) 029501

[6] S. Parrott, "Proof gap in "Sufficient conditions for uniqueness of the Weak Value by J. Dressel and A. N. Jordan, J. Phys. A **45** (2012) 015304, arXiv:1202.5604.

[7] Dressel J and Jordan A N Corrigendum: Sufficient conditions for uniqueness of the weak value, J. Phys. A: Math. Theor. **46** (2013) 029501