Essay
on
Restoring the quantum state after a measurement
by

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0 Disclaimer

This is an essay, not a research paper. Probably most of it is known to someone, though I've not seen much of it presented in the literature in accessible ways. There is a description of some original research without details, but the research was a failure in the sense that the results obtained were no better than known results.

1 Introduction

Several years ago there was a flurry of interest in

> "Reversal of the Weak Measurement of a Quantum State ..." by N. Katz, et al., Phys. Rev. Lett. **101** 200401 (2008).

It was selected as the subject of a "Viewpoint" article in the APS expository journal *Physics* [2]. The article is available online without a subsciption at physics.aps.org/articles/v1/34 . I will give some quotes from the article below, but the reader is urged to first read the original.

The Katz, et al., article describes an experiment in which a measurement on a quantum state was "undone", yielding a state identical to the one before the measurement. According to quantum theory as typically presented in older texts which consider only "projective" measurements, this should be impossible.

Both the original Katz, et al., article and the Viewpoint speak of the supposedly impossible restoration as having something to do with so-called "weak measurements" and erasure of information. Here are some quotes from the Viewpoint:

> "In quantum mechanics courses, students learn that the possible results of a quantum measurement of a physical quantity are the eigenvalues of the operator corresponding to the physical quantity. In other words, a measurement of the physical system "projects" it onto one of the eigenstates of this operator. In general, this only

> can happen in one direction: mathematically, the projection cannot be inverted, so it is an irreversible process. However, there are more gentle measurement schemes that only acquire partial information and so escape the constraint of traveling down this one-way street. A recent experiment on superconducting phase qubits performed by Nadav Katz and colleagues at University of California, Santa Barbara, and the University of California, Riverside [1], demonstrates that the effect of such a measurement can be "undone" and the initial state can be recovered."
>
> ...
>
> "It has long been understood that not every quantum measurement can be described by von Neumann's paradigm, which has come to be called the "collapse of the wave function" ... Recently, however, there has been much interest in a different kind of quantum measurement called "weak" measurement. The idea of weak (continuous) measurements was developed in quantum optics .... Although these measurements yield only limited information about the system, they allow a continuous observation that will perturb the system only weakly. The transition from the initial state of the system to the final state after the measurement due to the acquisition of information during the measurement does not correspond to a projection. As a result, the measurement can be inverted, and the initial state of the system can be recovered."
>
> ...
>
> "This surprising state recovery is (yet) another example that research on quantum computing and on experimental realizations of quantum bits leads to a better understanding of the foundations and the interpretation of quantum mechanics."

When I initially read this in 2008, it raised many questions which continued unanswered until now. I think I have found some of the answers, and hope to share them with the readers of this essay.

The following points will be made:

- The result of the experiment of Katz, et al., is an immediate consequence of the theory of "measurement operators". It is not surprising, if one believes the theory. It confirms part of the theory.

  This observation is not meant to deprecate the experiment, but merely to put it into context. I suspect that the context is a conceptual structure of "measurement operators" which has scarcely been experimentally tested, but I am not sufficiently familiar with the experimental literature to pose this as other than a suspicion. I would not be surprised to learn that the Katz, et al., experiment is the one of first (or even the very first) to test parts of the "measurement operator" theory.

- Both Katz, et al., and the Viewpoint article give me the impression that the restoration of the premeasurement quantum state from the postmeasurement is somehow conditional on the measurement being "weak". That seems to me to be misleading.

  The most common definition of a "weak" measurement is one which negligibly changes all states. This is generally made precise by introducing a "weak measurement" parameter which quantifies the "weakness" of the measurement. Under this definition, I question some of the statements of [1] and [2] seeming to relate "weakness" of the measurement to the possibility or probability of recovery of the premeasurement state.[1]

- In general, "weakness" is a sufficient condition for "undoing" or "reversing" a measurement with *some* nonzero probability, but is not a necessary condition. Measurements which greatly change the premeasurement state can sometimes be undone with probability 1.

- For the special case of *positive* measurement operators, "weakness" *is* correlated with the *probability* of reversal—the weaker the measurement, the greater the probability of reversal.

  The measurement operators of Katz, et al. [1] are sometimes positive, but not always. I have never seen the distinction between positive and general measurement operators made in a context in which "weakness" is claimed to be associated with reversibility.

- It is often suggested [1, 2, 4, 9] that successful reversal is somehow correlated with erasure of "information". This suggestion will be discussed in the context of a precise "trade-off" between reversibility and "information gain" for qubits (but not for higher dimensional systems) derived in [4]. The conclusion will be that the trade-off for qubits seems more a mathematical accident than a general principle to guide intuition.

In retrospect, the analysis leading to these general conclusions seems almost trivial both mathematically and physically. However, my limited reading in the field makes me wonder if the matter has often been viewed in a way that would expose the triviality.

2 Preliminaries

The following will assume that the reader is familiar with the mathematical structure of quantum mechanics as might be taught in any first course. Beyond

[1] During the preparation of this work, I have come across two papers, [4] and [9], which seem to use "weak measurement" in the entirely different sense of "non-projective measurement". So far as I know, this usage is not common. However, if this were adopted as the definition of "weak measurement", then some of the statements of Katz, et al. [1] and the Viewpoint article [2] could possibly be justified, though in a rather trivial way. It would be too confusing to analyze every statement twice in the context of two very different definitions, so I will take as the definition of "weak measurement" the one given in the main text — a measurement which negligibly changes all premeasurement states.

that, he[2] should be familiar with the concept of "measurement operators" as expounded, for example, in Nielsen and Chuang's text [3].

Most of the mathematics of the quantum mechanics of systems with a finite-dimensional state space is linear algebra. I am a mathematician with a specialty in operator theory, which is the generalization of linear algebra to Hilbert spaces of infinite dimension. I mention this to assure the reader that I am well qualified to handle the mathematics. I have chosen not to belabor the elementary mathematics, in favor of concentrating on the larger picture. If a reader is unsure of a mathematical step, he can be relatively sure that it will follow from routine application of basic concepts of linear algebra.

That is not meant to suggest that I am immune from mistakes. If a reader does spot an error, I would be grateful for notification.

When I learned quantum mechanics in the 1960's, the term "[quantum] measurement" was used differently than is common today. At that time the emphasis was on measuring observables like position and momentum that could take on a continuum of real values. Today the emphasis seems to be on observables like the spin of a quantum particle whose measurement yields only results from a discrete index set, which I will always take to be a finite set for expositional simplicity. Since the index set is finite, it may be taken to be the set $\{1, \ldots, n\}$ of consecutive positive integers, but the integers themselves have no physical significance (unlike, say, values of the momentum of a particle).

Below we shall consider the traditional formulation of quantum mechanics in which states of a system are represented by positive operators of trace 1 on a complex Hilbert space. We assume throughout that the Hilbert space is finite dimensional. Such states are sometimes called "mixed states" in traditional formulations. We use the common though slightly illogical convention that a "positive" operator is a Hermitian operator with non-negative eigenvalues. The operator is called "strictly positive" if all the eigenvalues are strictly positive.

2.1 Measurement operators

Today, measurement of a quantum system in a given state ρ is described by a collection $\{M_i\}_{i=1}^n$, of "measurement operators M_i satisfying .

$$\sum_i M_i^\dagger M_i = I \quad , \tag{1}$$

where I denotes the identity operator. The measurement yields two pieces of data:

1. a member k of the index set $\{1, \ldots, n\}$ which occurs with probability $\mathrm{Tr}[M_k \rho M_k^\dagger] = \mathrm{Tr}[M_k^\dagger M_k \rho]$, and

[2] Here and elsewhere, "he" is synonymous with "he or she". I follow the long-standing and sensible grammatical convention that when the antecedent of a pronoun is of unknown gender, either the male or the female pronoun carries the same meaning: "he", "she", "he or she", and "she or he" are synonymous.

2. a new quantum state
$$\sigma := \frac{M_k \rho M_k^\dagger}{\mathrm{Tr}[M_k \rho M_k^\dagger]} \quad , \tag{2}$$
called the *postmeasurement* state.

This assumes that $\mathrm{Tr}[M_k \rho M_k^\dagger] \neq 0$; otherwise the result k has probability zero. When all measurement operators are invertible, which is the only case we shall need to consider, all measurement results occur with positive probability, so for simplicity the exposition will ignore zero-probability cases.

The measurement replaces the premeasurement state ρ with the postmeasurement state σ. Since (mixed) states are represented by positive (necessarily Hermitian) operators of trace 1, the denominator of the fraction of the definition (2) of σ simply normalizes the trace of $M_k \rho M_k^\dagger$ to trace 1.

2.2 Does reversing a measurement entail "erasure" of "information"?

After a measurement, the premeasurement state no longer exists, but knowledge of the measurement result k gives us information about its relation to the postmeasurement state. There is no obvious way to erase this information. For example, we can write it down in indelible ink in a macroscopic lab notebook. It may be useful to keep this in mind when evaluating statements like some of those quoted above from the "Viewpoint" article or the following from [1]:

> "'In order for the [state-restoring] uncollapsing procedure to work, we have to erase the information that was already extracted classically. This distinguishes this measurement-induced uncollapsing from a 'quantum eraser' [ref. [10] of [3]], in which only potentially extractable information is erased."

2.3 Unnormalized states

It will sometimes be convenient to speak of "unnormalized states", by which we shall mean positive operators of positive trace not necessarily 1. An unnormalized state may always be "normalized" by dividing it by its trace to produce a state of trace 1.

2.4 The effect of measurements on pure states

A (mixed) state is called "pure" if it cannot be written nontrivially as a convex linear combination of other states. Recall that when we are using the usual picture of (mixed) states as positive operators of trace 1, pure states are are represented by projectors with one-dimensional range. For a nonzero vector s in the underlying (complex) Hilbert space, denote by P_s the projector on the one-dimensional subspace spanned by s. (For future reference, note that we need not and do not assume that s is normalized, since P_s as just defined is independent of the normalization.) Then after obtaining result k as described in subsection 2.1, the postmeasurement state σ is

$$\sigma = \frac{M_k P_s M_k^\dagger}{\mathrm{Tr}[M_k P_s M_k^\dagger]} \quad .$$

Since P_s has rank 1, so does $M_k P_s M_k^\dagger$ (except in the zero-probability cases which we are ignoring), and hence so does σ. By the spectral theorem, the only positive operators of rank 1 which have unit trace are projectors. Hence $\sigma = P_q$ for some nonzero vector q. Since the range of $\sigma = P_q$ is the range of $M_k P_s$, we must have q proportional to $M_k s$, and $\sigma = P_q = P_{M_k s}$.

In a common picture in which only pure states are considered and are represented by nonzero vectors up to normalization, the state change from premeasurement to postmeasurement is

$$\text{premeasurement } s \mapsto \text{postmeasurement } M_k s.$$

If we insist on normalizing pure states, then for a normalized s (i.e., $|s| := \sqrt{\langle s, s\rangle} = 1$), the above becomes

$$\text{premeasurement } s \mapsto \text{postmeasurement } \frac{M_k s}{|M_k s|} \quad .$$

When the measurement operators are pairwise orthogonal projectors, the measurement is called a *projective* measurement. That was the only kind I ever heard of back in the 1960's.

My impression is that the concept of "measurement operators" arose from the 1983 monograph [5] of Krauss; in the literature "measurement operators" are often called "Krauss operators". The text [3] states that the concept arose earlier:

> "The theory of generalized measurements which we have employed was developed between the 1940s and 1970s. Much of the history can be distilled from the book of Krauss." [5]

However, it does not give more detailed references, and I could not find any definite references to the origin of the "measurement operator" concept by browsing through the Krauss book.

When I learned of measurement operators around 2005, I imagined that the concept would be derivable from the structure of the 1960's quantum mechanics with which I was familiar. There is a questionable sense in which it can be, but only very recently have I come to realize that it is probably better to think of the concept of "measurement operators" as a genuine extension of quantum theory as commonly presented in texts, not only of the 1960's but also of much later vintage.

One reason can be seen immediately from the above discussion. Suppose a measurement $\{P_i\}$ is projective, i.e., the P_i are pairwise orthogonal projectors.

Suppose the result of a measurement is k, so for premeasurement state ρ , the postmeasurement state σ is

$$\sigma = \frac{P_k \rho P_k}{\mathrm{Tr}[P_k \rho P_k]} \quad .$$

Since $P_k^2 = P_k$ and also $P_i P_k = 0$ for $i \neq k$, if the measurement is performed again but with σ in place of ρ as premeasurement state, the postmeasurement state is again σ. In other words, performing a projective measurement twice yields the same result as performing it once; the second measurement does not change the state.

However, for arbitrary measurement operators $\{M_i\}$, the result of the second measurement need not be the same. This is easy to see when the initial state is pure and described by a vector s, so that the unnormalized state after the first measurement is $M_k s$, and after the second is $M_j M_k s$. For general measurement operators required to satisfy only the condition $\sum_i M_i^\dagger M_i = I$, there is no reason that $M_j M_k s$ should be proportional to $M_k s$. This seems an essential difference between projective measurements and general measurements.

3 "Undoing" a measurement

Now let us turn to the question of "undoing" a measurement with measurement operators $\{M_i\}$. "State restoration" and "state reversal" will be used synonymously for any procedure which converts a postmeasurement state back into the premeasurement state.

Suppose the premeasurement state is ρ, and the result of the measurement is k. We seek a second measurement with measurement operators $\{M'_i\}$ such that for some result j with nonzero probability, the postmeasurement state for the second measurement will be ρ:

$$\frac{M'_j M_k \rho M_k^\dagger M_j'^\dagger}{\mathrm{Tr}[M'_j M_k \rho M_k^\dagger M_j'^\dagger]} = \rho \quad . \tag{3}$$

This restoration will occur with probability

$$\frac{\mathrm{Tr}[M'_j M_k \rho M_k^\dagger M_j'^\dagger]}{\mathrm{Tr}[M_k \rho M_k^\dagger]} \quad .$$

We may choose notation so that $k = 1 = j$. If we want (3) to hold for all initial states ρ, then a sufficient[3] condition is that

$$M'_1 = c M_1^{-1} \tag{4}$$

[3]It seems likely that the condition is also necessary, but I have not examined this question because it will be peripheral to the discussion to follow. For (3) to hold for all pure states, it is necessary that (taking $k = 1 = j$ for simplicity) $M'_1 M_1 s = c(s)s$ for all vectors s, where $c(s)$ is a nonzero constant depending on s. If $c(s)$ can be shown to be independent of s, then (4) follows.

with c a nonzero scalar; that is, M'_1 is proportional to M_1^{-1}. The constant c cannot be too large because of the constraint

$$\sum_i {M'_i}^\dagger M'_i = I \quad ,$$

but otherwise can be arbitrary.

So, we can always restore any initial states ρ with nonzero probability so long as *at least one* of the original measurement operators $\{M_i\}$ is invertible. For a nontrivial projective measurement, none of the measurement operators are invertible.

Both the Katz, et al., and Viewpoint papers spoke of "weakness" of the measurement as somehow relevant. One definition of a "weak" measurement requires that *all* of the $M_i/||M_i||$ be close to the identity, so in particular, all are invertible. This is a sufficient condition for restoration with nonzero probability, but it is very far from a necessary condition.

This can be seen by considering the special case in which there is only one measurement operator, M_1, which must be unitary. A second measurement with only one measurement operator $M'_i := M_1^{-1}$ then reverses the original measurement with probability 1. Since M_1 can be an arbitrary unitary operator, it need not be close to the identity, so the measurement need not be "weak". In fact, given an initial pure state $\rho = P_s$, the pure postmeasurement state $M_1 \rho M_1^\dagger = P_{M_1 s}$ can be any pure state. If $M_1 s$ is orthogonal to s, the fidelity $\mathrm{Tr}[P_s P_{M_1 s}]$ between pre and post measurement states is as small as a fidelity can be, namely zero.

4 State restoration with an invertible measurement operator

4.1 A simple but nonoptimal protocol for state restoration.

Let us examine more closely the problem of restoration in case one of the measurement operators, say M_1, is invertible, assuming that the system Hilbert space is finite dimensional.[4] Let $\lambda_{\min}$ denote the smallest (necessarily positive) eigenvalue of $(M_1^\dagger M_1)^{1/2}$. Then $||M_1^{-1}|| = \lambda_{\min}^{-1}$.

We shall choose $M'_1 := cM_1^{-1}$ with c a positive constant chosen as large as possible (to assure the greatest probability of success with this simple method), which means that

$$c := \lambda_{\min} \quad .$$

Next define:

$$M'_2 := \sqrt{I - \lambda_{\min}^2 {M'_1}^\dagger M'_1} \quad .$$

[4]The assumption of finite dimensionality is only for expositional simplicity. What we shall say works equally well for infinite dimensional spaces with appropriate changes of language, e.g., replace "least eigenvalue of M" with "smallest point in the spectrum of M".

We shall show that any state ρ can be restored with probability p_{restore} at least

$$p_{\text{restore}} \geq \lambda_{\text{min}}^2 \quad . \tag{5}$$

For the particular method to be described, the restoration probability is exactly equal to λ_{min}^2, but it is not clear that other methods cannot do better.

Our protocol for restoration is the following.

1. Perform the measurement defined by $\{M_i\}$.
2. If the result is $i \neq 1$, declare failure.
3. If the result is $i = 1$, perform a second measurement defined by measurement operators $\{M_j'\}_{j=1}^2$ If the result is $j = 2$, declare failure.

The state will be restored if neither measurements results in failure. In that case, the final state will be the normalization to trace 1 of the unnormalized state

$$M_1' M_1 \rho M_1^\dagger {M_1'}^\dagger \propto M_1^{-1} M_1 \rho M_1^\dagger {M_1^{-1}}^\dagger \propto \rho \quad . \tag{6}$$

The λ_{min} factors have been omitted for clarity because any scalar multiple will be canceled after normalization. Since M_1' is proportional to M_1^{-1}, the left side of (6) is proportional to ρ and becomes ρ after normalization.

The probability that the first measurement succeeds is

$$p(\text{1st measurement succeeds}) = \text{Tr}[M_1 \rho M_1^\dagger] \quad .$$

The conditional probability that the second measurement succeeds given that the first did is

$$p(\text{2nd succeeds} \mid \text{1st succeeds}) = \text{Tr}\left[M_1' \frac{M_1 \rho M_1^\dagger}{\text{Tr}[M_1 \rho M_1^\dagger]} {M_1'}^\dagger \right] \quad .$$

Hence the probability of success for measurement 1 followed by success for measurement 2 is

$$\begin{aligned} p(\text{both measurements succeed}) &= p(\text{2nd succeeds} \mid \text{1st succeeds})\, p(\text{1st succeeds}) \\ &= \text{Tr}\left[M_1' \frac{M_1 \rho M_1^\dagger}{\text{Tr}[M_1 / \rho M_1^\dagger]} {M_1'}^\dagger \right] \text{Tr}[M_1 \rho M_1^\dagger] \\ &= \lambda_{\text{min}}^2 \text{Tr}[M_1^{-1} M_1 \rho M^\dagger {M_1^{-1}}^\dagger] \\ &= \lambda_{\text{min}}^2 \quad . \end{aligned} \tag{7}$$

It may seem surprising that the probability of restoration does not depend on the initial state ρ. Similar observations regarding both this independence and the formula (7) for the overall probability of reversal were made in an interesting paper of Cheong and Lee [4].

It should be emphasized that the probability calculated in (7) is the "overall" probability that the state reversal succeeds, not the conditional probability that state reversal succeeds given that the experimenter is presented with the state $M_1 \rho M_1^\dagger / \mathrm{Tr}[M_1 \rho M_1^\dagger]$ resulting from the initial measurement.

Note that nowhere was the "weakness" of the measurement used. One definition of "weak" measurement requires that all $M_i/||M_i||$ be close to the identity. Define $\lambda_{\max,i}$ to be the largest eigenvalue of $(M_i^\dagger M_i)^{1/2}$. For *positive* measurement operators, this definition is the same as requiring that

$$\frac{\lambda_{\max,i} - \lambda_{\min,i}}{\lambda_{\max,i}} = 1 - \frac{\lambda_{\min,i}}{\lambda_{\max,i}} \approx 0 \tag{8}$$

be small for all i. In the context of the experiment of Katz, et al. [1] in which measurement results other than 1 are ignored, only (8) for $i = 1$ might reasonably be required.

4.2 The simple protocol in the context of the experiment of Katz, et al. [1]

The simple state restoration protocol just described is essentially that employed in the experiment of Katz, et al. [1], though it is described differently there and there are some differences of detail. The measurement operators that they use,

$$M_1 := \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\phi_M}\sqrt{1-p} \end{bmatrix} \quad , \quad M_2^\dagger M_2 = I - M_1^\dagger M_1,$$

are not necessarily positive due to the factor $e^{-i\phi_M}$ in the $(2,2)$ entry. Here ϕ_M is described as "an accumulated phase due to an adiabatic change in the energy level spacing during the measurement", which I don't know how to interpret. Also, p represents a certain probability in their experiment which we have not discussed.

The only importance of ϕ_M for our considerations is that *if $e^{-i\phi_M}$ is not close to 1* (and it isn't in some cases, according to their formula (3)), then *the measurement is not weak for small p,* contrary to impressions given by [1] and [2]. A unit premeasurement vector state $\psi_{\text{before}} = \binom{a}{b}$ is transformed by a "successful" measurement (i.e. result 1) into

$$\psi_{\text{after}} = \frac{1}{N}\begin{pmatrix} a \\ e^{-i\phi_M} b\sqrt{1-p} \end{pmatrix}, \text{with the normalization factor } N := \sqrt{|a|^2 + (1-p)|b|^2}.$$

The fidelity (overlap) between the premeasurement and postmeasurement states is

$$|\langle \psi_{\text{before}} | \psi_{\text{after}} \rangle|^2 \quad = \quad \frac{(|a|^2 + |b|^2)^2 - [\; p|b|^2 + 2|a|^2|b|^2(1 - \cos\phi_M \sqrt{1-p}) \;]}{|a|^2 + (1-p)|b|^2},$$

which must be close to $1 = |a|^2 + |b|^2$ for a "weak" measurement which does not appreciably change the initial state ψ_{before}. Thus for small p, the measurement cannot be "weak" in this sense unless $\cos\phi_M \approx 1$.

Since the above criterion (8) for "weak" measurement applies only to positive measurement operators, let us specialize to the case $\cos\phi_M = 0$. Then a "weak" measurement corresponds to $p = 1-\lambda^2_{\min} \approx 0$, which is the same as (8) modified to apply to only the measurement operator M_1.

4.3 A better protocol for inverting a measurement

4.3.1 The protocol of Cheong and Lee

The method for inverting a measurement just given in the previous subsection is simple, but almost obviously not best possible. The protocol starts with a set of measurement operators $M_1, \ldots, M_n$, but can yield the desired inversion only if the first measurement happens to result in 1. If that measurement yielded result 2 through n, the simple protocol declares "failure", thus throwing away information obtained from the measurement. Instead of declaring "failure", one could obtain additional opportunities for inversion by proceeding in the same way as with initial outcome 1.

Define $\lambda_{\min,i}$ to be the smallest eigenvalue of $(M_i^\dagger M_i)^{1/2}$. For the inversion method just described, Cheong and Lee [4] note using reasoning similar to the above that the overall probability of restoration is

$$\text{probability of restoration using Cheong/Lee method } = \sum_{i=1}^{n} \lambda^2_{\min,i}. \qquad (9)$$

4.3.2 The Cheong/Lee protocol is best possible for "one-shot" inversion

For "one-shot" inversions (defined as requiring just one measurement to invert), the Cheong/Lee method is best possible. This is because

$$\begin{aligned} p(\text{restoration}) &= \sum_{i=1}^{n} p(\text{result } i \text{ followed by restoration}) \\ &= \sum_{i=1}^{n} \lambda^2_{\min,i} \quad . \end{aligned}$$

Here the probabilities in the sum of the first line are the "overall" probabilities of restoration when the result is i (i.e., the probability that the result is i *and* the state is restored), not the conditional probability of restoration given result i. The proof that the overall probability is $\lambda^2_{\min,i}$ is identical to the proof of subsection 4.1. The thing to note is that the overall probability derived there is an exact probability which is *independent of the initial state*, not simply a worst-case probability which might be better for some states. The argument just given would be invalid if $\lambda^2_{\min,i}$ were merely a worst-case probability.

As before, let $\lambda_{\min,i}$ denote the smallest eigenvalue of $(M_i^\dagger M_i)^{1/2}$, and $\lambda_{\max,i}$ the largest eigenvalue. For the case of just two measurement operators $M_1, M_2 =$

$\sqrt{I - M_1^2}$ operating on a space of dimension 2, we have $\lambda^2_{\min,2} = 1 - \lambda^2_{\max,1}$ (because $M_1^\dagger M_1 + M_2^\dagger M_2 = I$), and (9) reduces to

$$\text{overall probability of restoration} = 1 - \lambda^2_{\max,1} + \lambda^2_{\min,1} \quad . \tag{10}$$

4.3.3 Relation between "weakness" of the measurement and probability of restoration for positive measurement operators

Recall that for *positive* measurement operators, (8) related the relative sizes of $\lambda_{\min,i}$ and $\lambda_{\max,i}$ to the "weakness" of the measurement. Equation (10) relates the relative sizes of $\lambda_{\min,i}$ and $\lambda_{\max,i}$ to the probability of restoration, though not in precisely the same way as (8). Thus (8) and (10) together might be taken as a qualitative correlation between "weakness" for positive measurement operators and probability of restoration. A more precise relation will be developed below.

For positive measurement operators (only), one way to quantify the "weakness" of a measurement is to define a "weak measurement" parameter g by

$$g := \sum_{i=1}^{n} \left(1 - \frac{\lambda_{\min,i}}{\lambda_{\max,i}}\right) \quad , \tag{11}$$

so that values of g near 0 correspond to "weaker" measurements. Of course, other reasonable definitions of "weakness" of a measurement are possible. For example, a weak measurement parameter g' definied by

$$g' := \sum_{i=1}^{n} \lambda^2_{\max,i} \left(1 - \frac{\lambda_{\min,i}}{\lambda_{\max,i}}\right)$$

would allow a "weak" measurement to greatly alter some initial states if this occurred with sufficiently small probability.

If we do decide to quantify the "weakness" of a measurement by the parameter g of (11), then we can show that for positive measurement operators, a measurement is "weak" by this criterion if and only if the probability of restoration (9) is close to 1.

Before presenting the details, we remark that one should not read too much into this. First of all, the hypothesis that the measurement operators be positive is crucial, as the previous example of a single unitary measurement operator shows. The physical meaning of this assumption is not clear to me. Second, it is not true that the *possibility* of state restoration with nonzero probability is correlated in any way with "weakness" of the measurement. However, the possibility of state restoration *with probability near 1* does require a "weak" measurement (i.e., $g \approx 0$). Conversely, if the measurement is weak enough, (and continuing to assume that all measurement operators are positive) then the state can be restored with probability close to 1.

The details are as follows. From the algebraic identity $(1-x)^2 = (1-x)(1+x)$, one obtains the inequalities

$$\frac{1}{2}\sum_{i=1}^{n}\left(1 - \frac{\lambda^2_{\min,i}}{\lambda^2_{\max,i}}\right) \le g \le \sum_{i=1}^{n}\left(1 - \frac{\lambda^2_{\min,i}}{\lambda^2_{\max,i}}\right) \quad ,$$

from which it follows that $g \approx 0$ is equivalent to $\lambda_{\min,i}/\lambda_{\max,i} \approx 1$ for all i. Since $I = \sum_i M_i^\dagger M_i = \sum_i M_i^2$, for a Hilbert space of dimension d we also have

$$d\sum_{i=0}^{n} \lambda^2_{\min,i} \ \leq\ \mathrm{Tr}[I] \ = d \ \leq\ d\sum_i \lambda^2_{\max,i}, \quad \text{i.e.,} \tag{12}$$

$$\sum_{i=0}^{n} \lambda^2_{\min,i} \ \leq 1 \ \leq\ \sum_i \lambda^2_{\max,i} \quad . \tag{13}$$

From the equivalence of $g \approx 0$ with $\lambda_{\min,i}/\lambda_{\max,i} \approx 1$ just noted, it follows that $g \approx 0$ is equivalent to

$$\sum_i \lambda^2_{\min,i} \ = \ \sum_i \frac{\lambda^2_{\min,i}}{\lambda^2_{\max,i}} \lambda^2_{\max,i} \ \approx\ \sum_i \lambda^2_{\max,i} \quad .$$

Combining this with (13) shows that $g \approx 0$ is equivalent to

$$\sum_i \lambda^2_{\min,i} \approx 1 \quad . \tag{14}$$

This shows that for the Cheong/Lee method applied to *positive* measurement operators, $g \approx 0$ is equivalent to the probability of reversal being close to 1. This is the strongest rigorous embodiment that I know of a relation between weak measurement and state reversal.

4.4 Cheong and Lee's relation between "reversibility" and "information gain"

The interesting paper "Balance between information gain and reversibility in weak measurement" of Cheong and Lee [4] defines "reversibility" R of a measurement as in (9),

$$R := \sum_{i=1}^{n} \lambda^2_{\min,i} \quad , \tag{15}$$

and "information gain" as a rescaling of the quantity

$$G := \sum_{i=1}^{n} \lambda^2_{\max,i} \quad . \tag{16}$$

I am using G as a proxy for their "information gain" (which will be defined below) for simplicity of exposition. Also, I should make clear that these are not actually their *definitions*, but are calculated quantities which follow from their more primitive definitions. However, for a concise presentation it is convenient to start with (15) and (16) as definitions. The previous subsection explained why R is a natural definition of "reversibility" — it is the probability of reversal using the Cheong/Lee method, which is currently the best method known.

Their motivation for using G as the "information gain" is more complicated and will not be presented here. The idea, considerably oversimplified, is that if measurement result i occurs, then one's "best guess" at the initial state is the state which gives the maximal probability of obtaining result i. (This is similar to the motivation for the "maximum likelihood" method of statistical estimation.) Assuming for simplicity that the measurement operators are positive, the "best guess" will be the eigenstate ψ_{max} corresponding to the largest eigenvalue $\lambda_{\max,i}$ of the ith measurement operator.[5]

The confidence that we have in this guess will increase as $\lambda_{\max,i}$ increases. If $\lambda_{\max,i}$ is as small as it can be (i.e., equal to the smallest eigenvalue) then the measurement operator is a multiple of the identity, and when the result is i, the initial state is unchanged by the measurement. In that case, the measurement has given no usable information concerning the unknown initial state.

Thus we would expect a reasonable measure of the "information gain of the measurement to be a monotonic function of each of the $\lambda_{\max,i}$. Up to rescaling, any such function could serve, including G.

Of course, some such functions might have additional desirable properties. For example, by subtracting a suitable constant from G, we could obtain a measure of "information gain" which is zero for measurements which never change the state.

From the defining relation for positive measurement operators,

$$\sum_i M_i^\dagger M_i = \sum_i M_i^2 = I \quad ,$$

it follows that for a system with Hilbert space of dimension d,

$$d = \mathrm{Tr}[I] = \sum_i \text{ all squared eigenvalues of } M_i \geq \sum_i (\lambda^2_{\min,i} + \lambda^2_{\max,i}) = R + G. \tag{17}$$

Cheong and Lee [4] present a rescaled version of (17) as a "trade-off relation between reversibility and information gain", motivated by a more complicated argument. They call their "information gain" G_{max} and define it by

$$G_{max} := \frac{1}{d(d+1)}(d+G) \tag{18}$$

with d the dimension of the Hilbert space and G defined by as above in (16). Their rescaled version of our (17) (equation (17) of [4]) is

$$d(d+1)G_{max} + (d-1)R \leq 2d \quad . \tag{19}$$

However, the inequality in (19) or (17) seems a bit unsatisfying. For example, from (19) we cannot conclude that an increase in reversibility entails a necessary

[5] There is an ambiguity, not discussed by Cheong and Lee [4], in case the largest eigenvalue is degenerate, In that case, interpreting their mathematical expressions literally, the "best guess" may be taken to be any vector in the eigenspace corresponding to the largest eigenvalue.

reduction in information gain. Also the derivation of (17) shows clearly that the inequality will usually be strict for $d \geq 3$.

For the special case of $d = 2$ corresponding to qubits, the inequality in (19) becomes an equality, which shows a genuine tradeoff:

$$6\, G_{max} + R = 4 \quad . \tag{20}$$

In view of the way that this arises from (17), it appears more a mathematical accident which applies only to qubits than as a general principle to guide intuition.

4.5 A failed attempt at a better restoration method

We should remember that (9) applies only to one possible way of obtaining reversal, and it is not clear to me that more efficient methods are impossible. For example, one could consider the following protocol. Start with a measurement $\{M_1, M_2\}$ which one wants to reverse. Assume for simplicity that the M_i are positive. Starting with state ρ, try to reverse result 1 as described in the discussion leading to (5). If that fails, in the notation leading to (5), the system will be in unnormalized state $[I - (\lambda_{\min} M_1^{-1})^2]^{1/2}\, \rho\, [I - (\lambda_{\min} M_1^{-1})^2]^{1/2}$. Instead of giving up as before, try again to obtain the original state ρ. Though this will be impossible because $I - (\lambda_{\min} M_1^{-1})^2$ has a nontrivial nullspace, one could replace $\lambda_{\min} M_1^{-1}$ by a smaller operator $c\lambda_{\min} M_1^{-1}$ with $c < 1$ to obtain an additional opportunity for reversal. The procedure can be iterated to obtain an infinite sequence of reversal opportunities. Is it obvious to the reader that this cannot produce a better result than the simple method described after (5)?

Unfortunately, it does not seem to produce a better result than simply stopping after one try. According to my preliminary calculations, one does obtain an infinite sequence of reversal opportunities, but regardless of the choice of c, summing their probabilities produces *exactly* the same result as the original one-shot method. (In case this seems contradictory, note that the first try in the new method will also depend on c, so that the first term in the sum will be less than the probability of the original one-try method.) But I would be hesitant to bet that no clever person could produce a better method.

I have not been able to think of a way to see, *a priori*, that the final result of the expanded method will be independent of c and exactly equal to the result of the one-shot method which obtains an overall probability of reversal of $\lambda^2_{\min,i}$ when the measurement to be reversed yields result i. I cannot help wondering if some general, as yet undiscovered principle might be responsible.

I should also warn that the calculations leading to this conclusion have not been carefully written out and checked, so I cannot publicly guarantee their accuracy, and the reader should work them out for himself before relying on them. But privately I am sufficiently convinced that I have stopped working on this method.

5 Remark on "partial collapse" terminology

The older quantum mechanics, which considered only projective measurements, referred to the measurement process as "collapsing" the premeasurement state. Both Katz, et al., and the Viewpoint article refer to the postmeasurement state after a measurement with measurement operators as a "partial collapse". I do not think that this terminology is particularly helpful, even for projective measurements.

In the old way of thinking, the quantum state was an objective physical property of a system ("ontological" in current terminology), and a projective measurement was considered as replacing it by a different physical state. This replacement was called a "collapse", perhaps because there was no obvious way to undo it, as contrasted with unitary evolution described by the Schroedinger equation.

A recent thought-provoking article, "On quantum Theory" by B-G. Englert [7], takes a different view with which I basically agree:

> " 'Collapse of the state' or 'wave function collapse' are popular synonyms for state reduction. The connotation that the transition
>
> $$\left[\rho \mapsto \frac{M_k \rho M_k^\dagger}{\mathrm{Tr}[M_k \rho M_k^\dagger]}\right]$$
>
> is a dramatic dynamical process, as if the physical system were evolving, is clearly misleading. ... The statistical operator [ρ] is not a physical object, it *describes* [emphasis his] the object by encoding what we know about it, and state reduction is **the bookkeeping device for updating the description.** [emphasis mine] ... different physicists may very well use different, equally correct, statistical operators for predictions about [the same system]. "

In this quote, one can clearly see the influence of Bayesian ideas, even if one does not go so far as to consider the quantum state as a probability distribution over some inaccessible set of "ontological" (i.e., "physically real"] states. Though recent widely discussed arguments of Pusey, Barrett, and Rudolph [6, for an accessible introduction see [10]] suggest that quantum states are something more than a state of knowledge ("epistemic", in current jargon), that does not imply that a quantum state has nothing to do with a state of knowledge. Indeed, the very concept of the "statistical operator" of Englert's quote (a.k.a. "mixed state") involves a state of knowledge.

Englert views what some call "state collapse" as merely an updating of the "knowledge" portion of the state, and this seems to me a sensible view. If one takes it, the so-called "measurement problem" seems to vanish.

6 Afterword

On returning to familiar surroundings from a trip, I usually have the sense that I am not quite the same person I was before the trip. The same holds for intellectual trips. I have learned from writing this essay, and would probably write it slightly differently were I to start again. But my time is limited, and the audience uncertain, possibly nonexistent. Even were I to rewrite it, on completion I probably would feel the need for yet another rewriting. This afterword is a substitute for rewriting.

In the Introduction, I wrote:

> "The result of the experiment of Katz, et al., is an immediate consequence of the theory of 'measurement operators'. It is not surprising, if one believes the theory. It confirms part of the theory.
>
> This observation is not meant to deprecate the experiment, but merely to put it into context. I suspect that the context is a conceptual structure of "measurement operators" which has scarcely been experimentally tested, but I am not sufficiently familiar with the experimental literature to pose this as other than a suspicion. I would not be surprised to learn that the Katz, et al., experiment is the one of first (or even the very first) to test parts of the 'measurement operator' theory."

Since then, I have come across two relevant references, S. E. Ahnert and C. Page, "General implementation of all positive-operator-value measurements of single photon polarization states" [8], and [9].

The Ahnart/Page article describes how to physically implement any measurement operators on qubits. I have not read this in detail and cannot vouch for it, but I have no reason to doubt it.[6] Since it appears to use only noncontroversial quantum mechanics which has been around for over half a century, it seems good evidence that the "measurement operator" concepts are probably sound. Still, it seems strange that few experimental tests have been done, and that the only ones of which I know are quite recent.

The paper [9] of Chen, et al., entitled "Experimental test of the tradeoff relation in weak measurement" describes an experiment intended to verify the "trade-off" relation (20) of the Cheong/Lee paper [4] discussed earlier.[7] Recall that the Cheong/Lee paper presents what it describes as a "trade-off" between "reversibility" and information gain. As discussed earlier, there is a true tradeoff only for the special case of qubits. This is the setting of the [9] experiment, which does confirm the Cheong/Lee conclusion.

[6] I include this caveat only because the Physical Review journals, including the supposedly prestigious Phys. Rev. Lett., publish so many questionable papers. Refereeing standards seem to be minimal to effectively nonexistent. I try to avoid citing papers for which I cannot personally vouch.

[7] In [4] and [9], the term "weak measurement" is unequivocally used to describe a nonprojective measurement, as opposed to the more usual use of the term to describe a measurement which negligibly alters the premeasurement state.

This experiment was mainly of interest to me because it explicitly implements a measurement operator M_1 using a Sagnac interferometer. Nominally, M_1 is part of a measurement $\{M_1, M_2\}$, but in their setup either M_1 or M_2 can be implemented, but not at the same time. After M_1 "clicks", they then invert the output state using a second Sagnac interferometer, and check that it really was inverted using quantum tomography.

The only experiments known to me which test (even indirectly as does [9]) the usual "measurement operator theory are the experiment of Katz, et al. [1] and that of [9], both of which appeared within the past five years. It seems odd that experimental verification should be so hard to find for a theory which has been so extensively studied for at least thirty years (surely there must be thousands of theoretical papers on it).

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